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Remarks on the classification of wheeled mobile robots

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Abstract. The subject of this work is modeling and classification of single-bodied wheeled mobile robots (WMRs). In the past, it was shown that the kinematics of each such robot can be modeled by one out of only five different generic models. However, the precise conditions under which a model is the proper description of the kinematic capabilities of a robot were not clear. These shortcomings are eliminated in this work, leading to a simple procedure for model selection. Additionally, a thorough analysis of the kinematic models and a classification of their singularities are presented.

1 Introduction

WMR are typically developed with a specific application in mind so that the resulting design provides the level of mobility that is appropriate for the robot's operation. The design of the WMR implies its specific kinematics that is then used to derive and program the controller for the robot, which translates a desired movement into the appropriate actuation of the individual wheels (steering angles and rotational speeds). Once in operation, the control law and also higher level control layers, such as the path planner, will use the (inverse) kinematics implicitly through the implemented control algorithms. It is thus often impossible for such controllers to adapt their control laws once the kinematics of the drive changes significantly.

Such a change can occur in many different scenarios: one example is the case of a fault in the drive – e.g. an impaired steering actuator – which obviously has big influence on the kinematic capabilities of the robot. A fault-tolerant controller for a wheeled robot could be able to adapt to this new situation. Another example would be a mobile robot pushing a (passive) roll container. If the container has less degrees of freedom (DoF) than the robot, then the kinematic and dynamic model of the whole system need to be considered by the controller. For control of modular WMR like those presented in Hofbaur et al. (2010), automatic model selection is a prerequisite for choosing the proper controller for a specific wheel configuration. Furthermore, online reconfiguration might lead to significant changes in the kinematics. A similar issue arises for teams of mobile robots which are linked by holonomic constraints, possibly virtual or real ones, for example in collaborative transportation. Despite the fact that all these problems could be solved through a custom-designed controller that can account for specific and pre-defined operational situations, it would be desirable to handle such situations in a more general way. When applying model based control, the controller is either required to deduce a model for an operational situation online or select the proper model from a pre-defined ensemble and parametrize it (Gruber and Hofbaur, 2012). For wheeled mobile robots, the latter method is preferable, because Campion et al. (1996) have shown that there is a *minimal* model, still sufficient to describe the posture kinematics of a WMR, which can only take five different forms for all kinds of single-bodied wheeled mobile robots. They obtain this result by introducing conditions for non-degeneracy of WMR. The contribution of this work is a controllability-study that also results in such a classification, making the notion of non-degeneracy unnecessary. In existing works, the precise conditions for using one of these minimal models for some specific wheel configuration are not made clear. This work eliminates these shortcomings and also investigates the consequences of choosing a specific model.

Modeling of systems defined by wheel-like constraints is a well studied field, and most of the pioneering work was conducted in the last two decades of the last century. Early works are Alexander and Maddocks (1989) and Muir and Neuman (1987). A large step was the work of Campion et al. (1996), which showed – in addition to the existence of the above-mentioned “minimal” models – that WMR models are controllable, differentially flat and that robots with restricted mobility are non-holonomic. Furthermore, they can be transformed to chained form. These model properties were the onset for control design methodologies for tracking and stabilization controllers for WMR, and WMR are popular examples and benchmarks for controllers of non-holonomic systems (Morin and Samson, 2008). Also modeling of WMR is still an issue, and especially the growing interest in mobile manipulators gave rise to some issues related to the modeling and control of mobile robots with steering wheels only (Giordano et al., 2009; De Luca et al., 2010). One of the first publications related solely to the topic of modeling this kind of pseudo-omnidirectional robots was Betourne et al. (1996) and this work was supplemented by Thuilot et al. (1996). A differential geometric perspective on undulatory locomotion of wheeled robots is given in Ostrowski and Burdick (1998) or Bullo and Lewis (2005), who use the snakeboard as an example.

The classification of WMR models found by Campion et al. (1996) is reviewed in Sect. 2. Section 3 shows the demand for a precise understanding of the implications of choosing a certain model. In Sect. 4, we analyze the accessibility properties of the associated control system. Based on these structural properties, we are able to find conditions under which a kinematic model is the proper description for a robot. These conditions lead to different *modes of operation* for wheeled mobile robots, presented in Sect. 5. In this section, we further discuss the effects of neglecting sliding constraints in modeling and introduce a classification of WMR. A classification of singularities is suggested in Sect. 6. We show how our analysis is supported by results from differential geometry in Sect. 7.

2 Review

In this section, the classes of posture kinematic models of WMR are reviewed, as they are developed in Campion et al. (1996); Campion and Chung (2008). Only single-body-robots are considered, where wheels that roll without slipping are mounted over hinges to a single rigid body (the chassis). Types of wheels are Swedish wheels, casters, steerable standard and fixed standard wheels. Configuration variables of an unconstrained robot are elements of an l-dimensional configuration manifold Q, given by

$$Q = G \times M \tag{1}$$

$$= \mathrm{SE}(2) \times \mathbb{T}^{N_\mathrm{c}} \tag{2}$$

where G is the Special Euclidean Group SE(2) describing rigid motions (rotation and translations) in two-dimensional Euclidean space and M the manifold of all possible wheel orientations of center steerable wheels, with N_c the number of such wheels. An element of G is a robot pose $\boldsymbol{\xi}$, an element of M a steering angle configuration $\boldsymbol{\beta}_\mathrm{c}$. Accordingly, $\boldsymbol{q} = (\boldsymbol{\xi}^T\ \boldsymbol{\beta}_\mathrm{c}^T)^T \in Q$. We assume that standard wheels may not slide in a direction perpendicular to their rolling plane and each wheel touches the ground in a single point. This restriction is modeled by so-called sliding constraints, which introduce a dependence between the configuration variables of the robot. Sliding constraints for standard wheels are in the form of Pfaffian equations

$$\omega_i(\boldsymbol{q})\dot{\boldsymbol{q}} = 0. \tag{3}$$

In coordinates, for example, the sliding constraint of a standard wheel i is

$$[\cos(\alpha_i + \beta_i)\ \sin(\alpha_i + \beta_i)\ l_i \sin(\beta_i)]\mathbf{R}^T(\theta)\dot{\boldsymbol{\xi}} = 0$$

$$\boldsymbol{c}_i(\beta_i)\mathbf{R}^T(\theta)\dot{\boldsymbol{\xi}} = 0, \tag{4}$$

see Fig. 1. There, (l_i, α_i) are the polar coordinates of the contact point of wheel i in a robot-fixed reference frame $\mathbb{X}_1$, $\mathbb{X}_2$. The origin of the robot-fixed frame is located at the Cartesian coordinates (x, y) in the inertial frame $\mathbb{I}_1$, $\mathbb{I}_2$. The rotation between the robot-fixed and the inertial frame is described by the matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

For steered wheels, the co-vector fields ω_i depend on β_i and therefore on $\boldsymbol{q}$. The sliding constraints of center-steerable and fixed standard wheels, as formulated in Eq. (3), are Pfaffian equations, since they equate to zero. In contrast to that, the rolling constraints of all kinds of wheels and the sliding constraints of off-centered orientable wheels are equations with a right-hand-side that may be set via an input (steering or wheel drive). For this reason, we will assume that the sliding constraints of off-centered orientable wheels and rolling constraints in general do not impose restrictions of mobility and will therefore be neglected in the remainder.

Let r be the number of center-steerable and fixed standard wheels. The restriction of mobility for WMR originates from their corresponding r sliding constraints (Eq. 3). A motion $\dot{\boldsymbol{q}}$ is consistent with these sliding constraints, if it is an element of the annihilator (null-space) of their corresponding Pfaffians ω_i, $i \in \{1 \ldots r\}$. A basis of this annihilator spans a subspace of the tangential space $T_q Q$. This subspace is called a distribution $\Delta_\mathrm{c}(\boldsymbol{q})$, which can be constructed in the following way:

Arrange the co-vectors $\boldsymbol{c}_i(\boldsymbol{q})$ from Eq. (4) in so-called sliding constraint matrices $\mathbf{C}_{1f}$ and $\mathbf{C}_{1\mathrm{c}}$, corresponding to fixed standard wheels and steerable standard wheels, respectively, such that

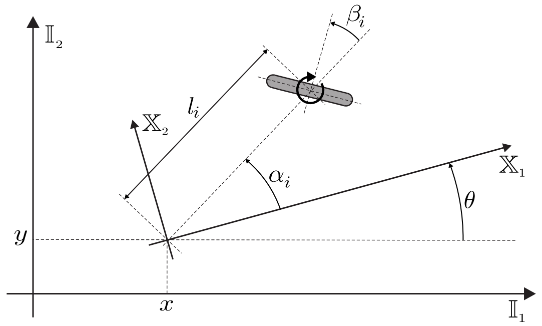

Figure 1. A center-steerable standard wheel.

$$\mathbf{C}_{1f}\mathbf{R}^T(\theta)\dot{\boldsymbol{\xi}} = 0$$
$$\mathbf{C}_{1c}\left(\boldsymbol{\beta}_c\right)\mathbf{R}^T(\theta)\dot{\boldsymbol{\xi}} = 0 \tag{6}$$

and

$$\mathbf{C}_1^*\left(\boldsymbol{\beta}_c\right) := \begin{pmatrix} \mathbf{C}_{1f} \\ \mathbf{C}_{1c}\left(\boldsymbol{\beta}_c\right) \end{pmatrix}. \tag{7}$$

A basis of the null-space of this matrix $\mathbf{C}_1^*(\boldsymbol{\beta}_c)$ spans the distribution Δ_c:

$$\mathrm{span}\left(\mathrm{col}\ \boldsymbol{\Sigma}\left(\boldsymbol{\beta}_c\right)\right) := \mathrm{null}\left(\mathbf{C}_1^*\left(\boldsymbol{\beta}_c\right)\right) \tag{8}$$
$$\Delta_c(\boldsymbol{q}) = \mathrm{span}\left(\mathrm{col}\ \mathbf{R}(\theta)\boldsymbol{\Sigma}\left(\boldsymbol{\beta}_c\right)\right). \tag{9}$$

This definition of $\Delta_c(\boldsymbol{q})$ follows the notation of Campion et al. (1996); Campion and Chung (2008) to illustrate that the null-space of $\mathbf{C}_1^*(\boldsymbol{\beta}_c)$ is the object that defines the dimension of the distribution. This is actually the case, since $\mathbf{R}(\theta)$ is of constant rank 3 and therefore has no influence on the dimension of $\Delta_c(\boldsymbol{q})$. Instead, this dimension depends on the steering angles $\boldsymbol{\beta}_c$ and is, by the rank-nullity theorem, given by

$$\delta_m\left(\boldsymbol{\beta}_c\right) = 3 - \mathrm{rank}\left(\mathbf{C}_1^*\left(\boldsymbol{\beta}_c\right)\right), \tag{10}$$

which is called *degree of mobility*. The *degree of steerability* is defined to be

$$\delta_s\left(\boldsymbol{\beta}_c\right) = \mathrm{rank}\left(\mathbf{C}_{1c}\left(\boldsymbol{\beta}_c\right)\right), \tag{11}$$

which is the number of independent steering actuators that have influence on the distribution.

According to the definition of Campion et al. (1996), a robot is *non-degenerate*, if

$$\mathrm{rank}\left(\mathbf{C}_{1f}\right) \le 1 \tag{12}$$
$$\forall \boldsymbol{\beta}_c \in M:\ \mathrm{rank}\left(\mathbf{C}_1^*\left(\boldsymbol{\beta}_c\right)\right) = \mathrm{rank}\left(\mathbf{C}_{1f}\right) + \mathrm{rank}\left(\mathbf{C}_{1c}\left(\boldsymbol{\beta}_c\right)\right) \tag{13}$$
$$\exists \boldsymbol{\beta}_c^*:\ \mathrm{rank}\left(\mathbf{C}_1^*\left(\boldsymbol{\beta}_c^*\right)\right) \le 2. \tag{14}$$

The numbers $\delta_m(\boldsymbol{\beta}_c)$, $\delta_s(\boldsymbol{\beta}_c)$ and the property of being either degenerate or non-degenerate allow the classification of *models* of WMR into five non-degenerate classes: (3, 0), (2, 0), (2, 1), (1, 1) and (1, 2) where each non-degenerate model class is identified by the notation (δ_m, δ_s). The sum of degree of mobility and degree of steerability is the number of overall DoF. Car-like robots (1, 1) and differential drive robots (2, 0) have 2 DoF, all other non-degenerate robots have 3 DoF. For every model class, Campion et al. provide a generic *posture kinematic model*, describing the posture kinematics of every model in a certain class. Each of these generic models is based on a specific choice of body-fixed frame (see Campion et al., 1996). These five posture kinematic models can be written in the form

$$\dot{\boldsymbol{x}} = \mathbf{G}(\boldsymbol{x})\boldsymbol{u} \tag{15}$$

with $\boldsymbol{x}$ the $3+\delta_s$-dimensional state vector, $\boldsymbol{u} = (\boldsymbol{\eta}^T\ \boldsymbol{\zeta}^T)^T$ the input vector and the matrix $\mathbf{G}(\boldsymbol{x})$ given by

$$\mathbf{G}(\boldsymbol{x}) = \mathbf{R}(\theta)\boldsymbol{\Sigma}\left(\boldsymbol{\beta}_c\right)$$
$$= \left(\boldsymbol{g}_{\eta_1}\cdots\boldsymbol{g}_{\eta_{\delta_m}}\boldsymbol{g}_{\zeta_1}\cdots\boldsymbol{g}_{\zeta_{\delta_s}}\right) \tag{16}$$

where $\boldsymbol{\eta}$ defines the δ_m wheel-speed-related inputs and $\boldsymbol{\zeta}$ the δ_s steering inputs.

3 Problem description

The models of WMR presented so far have some special features which are not addressed in popular publications like Campion et al. (1996), Canudas-de Wit et al. (1996), Siegwart and Nourbakhsh (2004), Campion and Chung (2008), etc. In contrast to these publications, the dependence of $\delta_m(\boldsymbol{\beta}_c)$, $\delta_s(\boldsymbol{\beta}_c)$ and $\Delta_c(\boldsymbol{q})$ on the steering angles $\boldsymbol{\beta}_c$ was explicitly pointed out in the previous section. This dependence already indicates that the mapping of a WMR to a certain model class is a *local* one. This means, that by the dependence of $\delta_m(\boldsymbol{\beta}_c)$ and $\delta_s(\boldsymbol{\beta}_c)$ on the steering angles $\boldsymbol{\beta}_c$, a specific robot can be mapped to multiple different classes according to its operational situation.

Recall that $\delta_m(\boldsymbol{\beta}_c)$ is the dimension of the constraint distribution Δ_c. Since this dimension changes with $\boldsymbol{\beta}_c$, Δ_c is no regular distribution on all of Q for all WMR. The mechanism which leads to a change in $\delta_m(\boldsymbol{\beta}_c)$ is the following: by setting specific steering angles $\boldsymbol{\beta}_c$, rows in $\mathbf{C}_1^*(\boldsymbol{\beta}_c)$ become linear dependent. As a result, the dimension of the null-space of $\mathbf{C}_1^*(\boldsymbol{\beta}_c)$ increases. Note that the set D of steering angles decreasing the rank of $\mathbf{C}_1^*(\boldsymbol{\beta}_c)$

$$D = \left\{\boldsymbol{\beta}_c \in M \,|\, \mathrm{rank}\left(\mathbf{C}_1^*\left(\boldsymbol{\beta}_c\right)\right) < \max\left(\mathrm{rank}\left(\mathbf{C}_1^*\left(\boldsymbol{\beta}_c\right)\right)\right)\right\} \tag{17}$$

is nowhere dense in M. The occurrence of such nowhere dense sets and their relation to the model types is illustrated in the following example:

Example 1 *Consider a WMR with two center-orientable standard wheels as shown in Fig. 2a and b. In the configuration shown in Fig. 2a, the wheel axes intersect in a* *point defining the Instantaneous Center of Rotation (ICR). Another steering configuration is shown in Fig. 2b. In this configuration, the wheel axes coincide and the rank of the annihilator of the constraint distribution drops from two to one: the constraint distribution Δ_c has constant dimension one on the open dense subset*

$$Q_1 = Q \setminus Q_2, \tag{18}$$

consisting of configurations similar to the one shown in Fig. 2a, and dimension two on

$$Q_2 = \{(x, y, \theta, \beta_1, \beta_2) \,|\, \beta_1 \in \{0, \pi\} \text{ and } \beta_2 \in \{0, \pi\}\}, \tag{19}$$

which contains four disjoint submanifolds representing steering configurations similar to those shown in Fig. 2b. These special configurations are instances of what will be called singularity of type A in the remainder. Following the procedure from Campion et al. (1996), reviewed in Sect. 2, one obtains

$$\delta_m = 1,\ \delta_s = 2 \text{ on } Q_1 \tag{20}$$

$$\delta_m = 2,\ \delta_s = 0 \text{ on } Q_2. \tag{21}$$

This illustrates the configuration dependency of the numbers (δ_m, δ_s) *and* Δ_c *is not a regular distribution.*

Example 1 shows that the dimension of the distribution Δ_c is not constant on all of Q. Intuitively, this is in contradiction with the configuration-independent number of DoFs. However, this is not the case. The point is that on Q_1 (wheel axes not aligned), the ICR is fixed by the intersection of the wheels axes. The location of this ICR is fixed via *two* steering inputs, while the tangential velocity is set via *one* wheel speed. The speed of the other wheel cannot be freely chosen but needs to be consistent with its rolling constraint. However, on Q_2, the sliding constraints only restrict the ICR to lie on the common axis through both wheels. The *two* wheel speeds define the location of the ICR on this axis via the rolling constraints. As soon as a steering angle is changed, the configuration leaves Q_2.

In the set Q_2, kinematic constraints of the robot become dependent, increasing its number of directly accessible degrees of freedom δ_m. This fact can be asserted in two simple thought experiments: think of a passive robot with two center-steerable standard wheels. In the first experiment, wheels are oriented as in Fig. 2a and the robot is pushed manually in a direction while the steering angles are held constant. Then the ICR remains at a fixed point and the motion of the robot describes a circle around this ICR. For the second experiment, change the steering angles such that the wheel axes coincide (the rolling planes are parallel, see Fig. 2b). In this situation, the robot can be pushed to any desired position without steering the wheels but rotating the whole robot.

To the best of our knowledge, such a property was not mentioned in literature and shows a demand for a deeper understanding of the relation between robots and models.

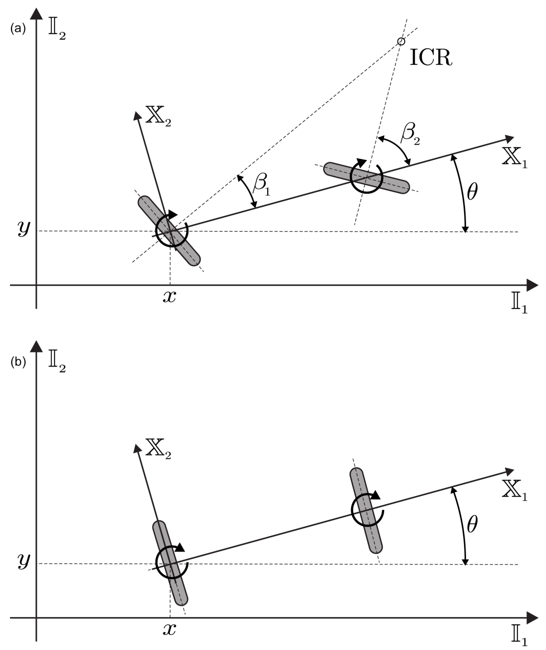

Figure 2. **(a)** A WMR with two center-steerable standard wheels in a non-singular configuration. **(b)** A WMR with two center-steerable standard wheels in a singular configuration.

4 Accessibility

Any robot which is at least capable of reaching any pose in the plane is able to fulfill most of the typical tasks of WMR. From a control engineering point of view, this formulation already sounds familiar, since accessibility is a well-known concept in the field of controllability analysis of nonlinear control systems.

Recall from Sect. 2, that the vector fields that span the nullspace of $\mathbf{C}_1^*(\boldsymbol{\beta}_c)$ span the distribution $\Delta_c(\boldsymbol{q})$, and a robot is able to move along these vector fields. To account for steering motions, this distribution $\Delta_c(\boldsymbol{q})$ is extended by the corresponding number of dimensions and vector fields modeling these steering motions to obtain a distribution $\Delta(\boldsymbol{q})$. Even though it seems like $\Delta(\boldsymbol{q})$ should now contain all the motions consistent with the sliding constraints, this is actually not the case. Nonlinear systems with multiple inputs, like WMR, are not limited to move along these vector fields in $\Delta(\boldsymbol{\beta}_c)$. By consecutive execution of vector field motions (e.g. steer-drive-steer-drive), "new" motion directions might result. A motion along such a "new" direction, a so-called Lie bracket motion, is described by a vector field $\boldsymbol{g}_m$, obtained by applying the Lie bracket operation on two vector fields $\boldsymbol{g}_k, \boldsymbol{g}_l \in \Delta$:

$$\boldsymbol{g}_m = [\boldsymbol{g}_k, \boldsymbol{g}_l] = \frac{\partial \boldsymbol{g}_l}{\partial \boldsymbol{q}} \boldsymbol{g}_k - \frac{\partial \boldsymbol{g}_k}{\partial \boldsymbol{q}} \boldsymbol{g}_l. \tag{22}$$

The involutive closure of the distribution Δ is denoted $\mathrm{inv}(\Delta)$. It is a distribution spanned by the vector fields in Δ and a minimal subset of Lie brackets of these vector fields, such that all remaining Lie brackets lie in inv(Δ). This involutive closure of the distribution Δ is finally the object, that is truly capable of describing all the motions that are consistent with the sliding constraints of a WMR (this holds because the vector fields that span Δ are analytic, see e.g. Isidori, 1995).

The set of poses accessible by motions in inv(Δ) is found by applying Chow's Theorem (Choset et al., 2005; Bullo and Lewis, 2005): if

$$T_{\xi}G \subset \mathrm{inv}(\Delta(\boldsymbol{q})) \tag{23}$$

over a trajectory (integral curve) $\boldsymbol{q}(t)$, $\forall t \in [0,\ T]$, then all poses $\boldsymbol{\xi}_T \in G$ are accessible from any initial configuration $\boldsymbol{q}_0$. Remarkably, this holds not just for the tangent space $T_{\xi}G$, but for the whole tangent bundle TG: the dependence of $\Delta(\boldsymbol{q})$ on $\boldsymbol{\xi}$ originates from the rotation matrix in Eq. (9). However, such a rotation does not influence the dimension of the space that is spanned by the distribution. As a result, condition Eq. (23) can be given in the more general, pose-independent form:

$$TG \subset \mathrm{inv}\left(\Delta\left(\boldsymbol{\beta}_{\mathrm{c}}\right)\right). \tag{24}$$

Accordingly, in order to show that every pose $\boldsymbol{\xi} \in G$ is accessible, one only has to find a single steering configuration for which $\mathrm{inv}(\Delta(\boldsymbol{\beta}_{\mathrm{c}}))$ spans TG. However, this is still a somewhat abstract condition. For this reason, we give an equivalent condition in the following theorem:

Theorem 1 *A WMR is able to reach any pose in the obstacle-free plane, if and only if all its fixed wheels are co-aligned, that is,*

$$\mathrm{rank}\left(\mathbf{C}_{1f}\right) \leq 1. \tag{25}$$

Proof:
Necessity: if this condition is violated, then Δ contains only one single vector field ($\mathrm{rank}(\mathbf{C}_{1f}) = 2$) or is empty ($\mathrm{rank}(\mathbf{C}_{1f}) = 3$). In the latter case, the robot is not able to move at all. In the former case, the distribution is trivially involutive and thus not able to span a space of dimension larger than 1, but $\dim(T_{\xi}G) = 3$. This means that when $\mathrm{rank}(\mathbf{C}_{1f}) = 2$, the robot is only capable of moving around a fixed ICR, that is, on a straight line or circle and therefore not able to reach any desired pose in the plane.

For sufficiency, we have to show that every robot with $\mathrm{rank}(\mathbf{C}_{1f}) \leq 1$ is able to reach any desired pose. The rank of the sliding constraint matrix $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}))$ can take values from zero to three. Let us analyze these valuations case-by-case:

1. $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}})) = 0$: the distribution spanned by the nullspace of $\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}})$ is by itself three-dimensional, involutive and spans $T_{\xi}G$. Due to Eq. (24), every pose in the plane is reachable.

2. $\exists \boldsymbol{\beta}_{\mathrm{c}}^*$: $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}^*)) = 1$: the nullspace of $\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}^*)$ has dimension two. Let the two vector fields, that span this nullspace be $\boldsymbol{g}_{\eta_1}(\boldsymbol{\beta}_{\mathrm{c}}^*)$ and $\boldsymbol{g}_{\eta_2}(\boldsymbol{\beta}_{\mathrm{c}}^*)$, i.e. $\Delta_{\mathrm{c}} = \{\boldsymbol{g}_{\eta_1},\ \boldsymbol{g}_{\eta_2}\}$ and $\dim(\Delta_{\mathrm{c}}) = 2$. The Lie bracket $[\boldsymbol{g}_{\eta_1}(\boldsymbol{\beta}_{\mathrm{c}}^*),\ \boldsymbol{g}_{\eta_2}(\boldsymbol{\beta}_{\mathrm{c}}^*)]$ does not lie in the space spanned by $\Delta_{\mathrm{c}}(\boldsymbol{\beta}_{\mathrm{c}}^*)$ and thereby gives the third direction, such that $TG \subset \mathrm{inv}(\Delta_{\mathrm{c}}(\boldsymbol{\beta}_{\mathrm{c}}^*))$ and the robot is able to reach any pose.

   Note that for $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}^*))$ to be one, the axes of all wheels – if there are more than one – must be co-aligned.

3. $\exists \boldsymbol{\beta}_{\mathrm{c}}^*$: $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}^*)) = 2$: the nullspace of $\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}^*)$ has dimension one. Let the vector field, that spans this nullspace be $\boldsymbol{g}_{\eta_1}(\boldsymbol{\beta}_{\mathrm{c}}^*)$. At all but a nowhere dense set S of singularities (these singularities are treated in more detail in Sect. 7), this vector field depends at least on one steering angle, because otherwise $\mathrm{rank}(\mathbf{C}_{1f})$ would be two. Construct Δ by extending Δ_{c} about a dimension and a steering vector field $\boldsymbol{g}_{\zeta_1}$. Build the following Lie brackets to obtain inv(Δ): $\mathrm{inv}(\Delta) = \{\boldsymbol{g}_{\eta_1},\ \boldsymbol{g}_{\zeta_1},\ [\boldsymbol{g}_{\eta_1},\ \boldsymbol{g}_{\zeta_1}],\ [\boldsymbol{g}_{\eta_1},\ [\boldsymbol{g}_{\eta_1},\ \boldsymbol{g}_{\zeta_1}]]\}$. Again, we get the result that $TG \subset \mathrm{inv}(\Delta_{\mathrm{c}}(\boldsymbol{\beta}_{\mathrm{c}}^*))$, allowing to conclude that the robot is able to reach any pose.

   Note that for $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}))$ to be two the axes of all wheels must intersect in one single point, the ICR.

4. $\exists \boldsymbol{\beta}_{\mathrm{c}}^*$: $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}^*)) = 3$: in this case, the nullspace of $\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}^*)$ is empty. However, the fixed wheels only contribute one to this rank of three. This means that the remaining rank of two must originate from the center-steered wheels. In other words, for steering angle configurations in

   $$B_3 = \left\{\boldsymbol{\beta}_{\mathrm{c}} \in M | \mathrm{rank}\left(\mathbf{C}_1^*\left(\boldsymbol{\beta}_{\mathrm{c}}\right)\right) = 3\right\}, \tag{26}$$

   the axes of the wheels do not intersect in one point. However, since all but the co-aligned fixed wheels are steerable, it must be possible to steer the wheels such that all axes intersect in one point (see Sect. 7 for details). Summing up, this means that every robot with a steering angle configuration $\boldsymbol{\beta}_{\mathrm{c}}^*$ such that $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}^*)) = 3$ must have another steering angle configuration $\boldsymbol{\beta}_{\mathrm{c}}^+$ with $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}^+)) = 2$ if condition (Eq. 25) holds. ■

An immediate consequence of Theorem 1 is the classification discovered by Campion et al. (1996): $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}}))$ can either be 0, 1 or 2, giving degrees of mobility δ_{m} of 3, 2 or 1, respectively. The number of independent steering degrees of freedom δ_{s} is given by $\mathrm{rank}(\mathbf{C}_1^*(\boldsymbol{\beta}_{\mathrm{c}})) - \mathrm{rank}(\mathbf{C}_{1f})$, which can also only take values 0, 1, 2. Table 1 shows inv(Δ) for all model classes, where vector fields that span TG are underlined. The Degree of Nonholonomy (DoN), also shown in the table, is the highest degree of the Lie-brackets in inv(Δ).

There is a slight difference between the set of robots considered here, and the set considered by Campion et al. (1996): in their work, some robots with pathological wheel distributions like for example the one shown in Fig. 3

Table 1. How inv(Δ) spans TG. Vector fields and Lie brackets that span TG are underlined.

$(\delta_\mathrm{m}, \delta_\mathrm{s})$	inv(Δ)	DoN	DoF
(3, 0)	$\underline{\boldsymbol{g}_{\eta_1}}, \underline{\boldsymbol{g}_{\eta_2}}, \underline{\boldsymbol{g}_{\eta_3}}$	1	3
(2 ,1)	$\underline{\boldsymbol{g}_{\eta_1}}, \underline{\boldsymbol{g}_{\eta_2}}, \boldsymbol{g}_{\zeta_1}, \underline{[\boldsymbol{g}_{\eta_1}, \boldsymbol{g}_{\eta_2}]}$	2	3
(1, 2)	$\underline{\boldsymbol{g}_{\eta_1}}, \boldsymbol{g}_{\zeta_1}, \boldsymbol{g}_{\zeta_2}, \underline{[\boldsymbol{g}_{\eta_1}, \boldsymbol{g}_{\zeta_1}]}, \underline{[\boldsymbol{g}_{\eta_1}, \boldsymbol{g}_{\zeta_2}]}$	2	3
(2, 0)	$\underline{\boldsymbol{g}_{\eta_1}}, \underline{\boldsymbol{g}_{\eta_2}}, \underline{[\boldsymbol{g}_{\eta_1}, \boldsymbol{g}_{\eta_2}]}$	2	2
(1, 1)	$\underline{\boldsymbol{g}_{\eta_1}}, \boldsymbol{g}_{\zeta_1}, \underline{[\boldsymbol{g}_{\eta_1}, \boldsymbol{g}_{\zeta_1}]}, \underline{[\boldsymbol{g}_{\eta_1}, [\boldsymbol{g}_{\eta_1}, \boldsymbol{g}_{\zeta_1}]]}$	3	2

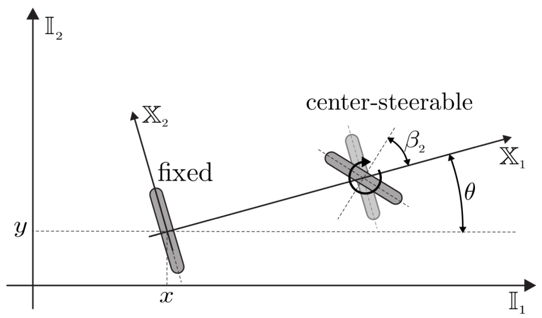

Figure 3. A robot with one fixed and one center-steerable wheel, that would be degenerate in the sense of Campion et al. (1996), although it is still able to reach any posture in the plane.

are defined to be degenerate. This is seen in Eq. (13): the robot has one fixed wheel, so $\mathrm{rank}(\mathbf{C}_{1f}) = 1$ and one steerable wheel, giving $\mathrm{rank}(\mathbf{C}_{1\mathrm{c}}(\boldsymbol{\beta}_\mathrm{c})) = 1 \forall \boldsymbol{\beta}_\mathrm{c}$. However, there exists a steering angle configuration $\hat{\boldsymbol{\beta}}_\mathrm{c}$ such that $\mathrm{rank}(\mathbf{C}_1^*(\hat{\boldsymbol{\beta}}_\mathrm{c})) = 1 \neq \mathrm{rank}(\mathbf{C}_{1f}) + \mathrm{rank}(\mathbf{C}_{1\mathrm{c}}(\hat{\boldsymbol{\beta}}_\mathrm{c}))$. Equation (13) does therefore not hold for all $\boldsymbol{\beta}_\mathrm{c}$. In our work, we want to include such robots as they may appear as a result of reconfiguration of modular WMRs (Mutambara, 1998; Hofbaur et al., 2010) or faults and are still able to reach any pose in the plane.

There is another difference in the approach of Campion et al. (1996) and Theorem 1: Campion et al. (1996) first define conditions for non-degeneracy, from which they obtain the classification. Then they show that non-degenerate robots are controllable. In the approach taken in this work, the classification follows from a controllability argument

We will therefore avoid using the notation of degeneracy and instead simply distinguish between wheel configurations that are controllable or not.

5 Classification

When talking about wheel (hardware) configurations and model classes, it is impractical to use the same notation $(\delta_\mathrm{m}, \delta_\mathrm{s})$ for both. Therefore, in contrast to Campion et al. (1996), we do not associate sets of wheel (hardware) configurations to the pair $(\delta_\mathrm{m}, \delta_\mathrm{s})$. Instead, we interpret $(\delta_\mathrm{m}, \delta_\mathrm{s})$ as a model

Table 2. The relation of hardware designs and modes of operation.

MoO↓ HW-type→	I	III	II	Va	Vb	IV
(3, 0)	r	n	–	n	n	–
(2, 1)	v	r	–	n	n	–
(2, 0)	v	v	r	n	r	n
(1, 2)	v	v	–	r	r	–
(1, 1)	v	v	v	v	v	r
Number of Swedish wheels	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0
Number of oc-steerable wheels	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0	≥ 0
Number of c-steerable wheels	0	1	0	≥ 3	≥ 2[a]	≥ 1
Number of fixed[b] wheels	0	0	≥ 1	0	0	≥ 1
Singularities in MoO (1, 2)	–	–	–	B, C	A, C	–

[a] The contact points of all center-steerable wheels lie on a straight line. [b] fixed standard wheels have to share a common axis. If steered standard wheels are placed on this axis, they must be counted as fixed wheels with similar orientation as the fixed wheels.

class or "mode of operation" (MoO) and introduce a separate notation for hardware types in this section.

Controllable robots may have multiple structurally different controllable wheel configurations. This fact was illustrated in Example 1, where the robot was found to have two disjoint sets of controllable wheel configurations. The property of the singular set Q_2 of being *nowhere dense* suggests the following method to find a configuration-independent classification: (1) partition Q into sets Q_i where $\delta_\mathrm{m}(q_i)$ and $\delta_\mathrm{s}(q_i)$ is constant $\forall q_i \in Q_i$. (2) Choose the only set Q_i which is dense in Q. However, this approach fails for robots with more than two center-steerable standard wheels: in this case the only dense set Q_i corresponds to degenerate configurations, since more than two arbitrarily oriented standard wheels will block any motion.

Previous works Thuilot et al. (1996), Betourne et al. (1996) and Giordano et al. (2009) already deal with robots with more than two center-steerable or fixed standard wheels. However, they treat this topic in substantially different ways: Thuilot et al. (1996) and Betourne et al. (1996) represent all these robots with (1, 2) models, whereas Giordano et al. (2009) use (2, 1).

To resolve this ambiguity the following definition is made:

Definition 1 *A WMR is said to operate in mode* $(\bar{\delta}_\mathrm{m}, \bar{\delta}_\mathrm{s})$ *in the time interval* $[t_0, t_0 + T]$, *if it moves along a trajectory (integral curve)* $\boldsymbol{q}(t)$ *for which* $\delta_\mathrm{m}(\boldsymbol{q}) = \bar{\delta}_\mathrm{m} = const.$ *and* $\delta_\mathrm{s}(\boldsymbol{q}) = \bar{\delta}_\mathrm{s} = const.$ $\forall t \in [t_0, t_0 + T]$.

When a robot operates in a mode $(\bar{\delta}_\mathrm{m}, \bar{\delta}_\mathrm{s})$ at some time instant t, then the corresponding model is the proper description of its posture kinematics at that time. In this context, the following question comes to mind: which WMR hardware designs are able to operate in which modes of operation? The answer to this question is given in Table 2, which is read in the following way: sum up the number of wheels of a specific type, then search the column that matches. Watch the coalignment conditions from footnotes a and b. These wheel counts determine the hardware type. Individual Swedish and off-center steerable wheels in drives that combine different types of wheels do not lead to a restriction of mobility. Accordingly, they have no influence on the hardware type. The upper part of the table then shows the possible modes of operation for the hardware type. Cells containing an "r" mark a Mode of Operation (MoO) resulting from the restriction due to physical sliding constraints. We call these the *physical* modes. The modes marked by "v" can be enforced by additional *virtual* constraints. If a mode is labeled with "n", this means that by driving the robot in this mode, physical sliding constraints of steerable wheels are neglected. With *neglected*, we mean that not all physical sliding constraints are represented in the model. The effect of such a concept is that for sufficiently fast actuators and motions with sufficiently low accelerations all sliding constraints are respected, while for motions with higher accelerations or slower actuators one or more sliding constraints will be violated and the robot will slip. Details are discussed in Sect. 5.3.

The hardware design-types that we introduce in Table 2 are equivalence classes of hardware designs under the following equivalence relation: $R_1 \sim R_2 \Leftrightarrow$ Robot R_1 and robot R_2 are able to operate in the same set of modes of operation without neglecting sliding constraints.

Example 2 *Let R_1 be a robot with two center-steerable wheels and R_2 a robot with two co-aligned fixed wheels. $R_1 \nsim R_2$, because R_1 is able to operate in mode (1, 2), (2, 0) (cf. Example 1) and even in mode (1, 1), whereas R_2 is only able to operate in modes (2, 0) and (1, 1).*

Example 3 *Let R_1 be a car with four (two steered and two co-aligned fixed) wheels and R_2 a motorcycle. R_1 and R_2 do not belong to the same class, because both are able to operate in mode (1, 1) only.*

The robot type numbers were chosen to match the classification of two- and three-wheeled robot classification from Gracia and Tornero (2007).

The last row in Table 2 shows which kinds of singularities must be considered, when the corresponding hardware-type is operating in mode (1, 2).

5.1 A HW-type Vb-robot operating in different modes

Exemplary, the following description shows how a robot with two or more coaligned center-steerable standard wheels (HW-type Vb) can operate in different modes. An example for a HW-type Vb robot is the snakeboard (Ostrowski and Burdick, 1998), typically operating in mode (1, 2).

(2, 0): a HW-type-Vb robot operates in mode (2, 0), if the axes of all wheels are co-aligned. In such a configuration, the rank of $\mathbf{C}_{1c}(\boldsymbol{\beta}_c)$ is one, which gives $\delta_m = 2$. This is equivalent to forcing the ICR to lie on the common axis of all wheels. Due to this co-alignment-condition there is no freedom in choosing $\boldsymbol{\beta}_c$, resulting in $\delta_s = 0$.

(1, 2): a robot is able to operate in this mode if the ICR is defined by a unique intersection of the wheel axes. If these axes coincide, then this condition is not satisfied and the robot is forced to operate in mode (2, 0). This means that, in order to operate in mode (1, 2), a robot must not reach a configuration that is consistent with the conditions of operating in mode (2, 0). Thus, for HW-type-Vb robots, $Q_{(2,0)}$ appears as set of singularities in $Q_{(1,2)}$. This is a serious issue, as illustrated in the following example:

Example 4 *Figure 6a and b illustrates a HW-type Vb WMR with two center-orientable wheels moving along a circular path. In Fig. 6a, the orientation of the robot frame is constant relative to a Frenet frame. The steering angles $\boldsymbol{\beta}_c$ are constant and the robot is able to follow the circular path in mode of operation (1, 2). In Fig. 6b, the orientation of the robot frame is held constant relative to an inertial frame. In this case, the steering angles $\boldsymbol{\beta}_c$ are not constant and the robot is not able to follow the circular path in mode of operation (1, 2). At the time instant shown in the upper right, the ICR lies on the straight line through both wheel contact points. This corresponds to mode of operation (2, 0). Along every circular path with constant orientation, there will be two configurations where the ICR is required to lie on the line through the contact points of both wheels, which is a configuration in $Q_{(2,0)}$.*

(1, 1): mode (1, 1) can be enforced for a HW-type-Vb robot by introducing a virtual constraint that blocks the steering of exactly one wheel. The axis of this blocked wheel must not coincide with the wheel contact points of the other wheels, otherwise forcing mode (2, 0).

(3, 0): by driving a HW-type-Vb robot in this mode, the restrictions from the sliding constraints are neglected for all wheels.

(2, 1): when a (2, 1) model is used for a HW-type-Vb robot, the sliding constraints are neglected for all but one wheel.

Due to its *two* different "physical modes", HW-type Vb is the most complex hardware type. For the other hardware types, finding the physically enforced modes of operation is as simple as determining δ_m and δ_s for an arbitrary configuration that allows motion.

5.2 Virtually constrained modes

For the virtually constrained modes respecting all sliding constraints, the following conditions must be satisfied:

1. the number of DoF of the desired mode must be lower or equal to the number of DoF of the physical mode;

2. the DoN of the desired mode must be higher or equal to the DoN of the physical mode (see also Table 1).

While the necessity of the first condition is obvious, the second one might require explanation. In the previous section, Lie-bracket motions were introduced as directions reachable by consecutive motions along different vector fields (e.g. "drive-steer-drive"). So, while some robots are able to drive directly in a certain direction, others may require Lie-bracket motions to do so. For this reason, Lie-bracket motions can be considered slower than vector-field motions. The higher the order of the Lie-bracket, the slower the motion. A WMR is therefore only able to drive in a mode involving Lie-brackets of equal or higher order than its own physical mode.

5.3 The effects of neglecting constraints

The mappings from Table 2 marked with "r" or "v" provide accurate models for each hardware type. This is to be understood in the following way: if a model for a mode of operation is selected according to the classification from Table 2, then this model captures all the restrictions of mobility of the WMR originating from its sliding constraints. The modes marked with "n" in Table 2 do not contain all of these restrictions. Especially for hardware-types Va and Vb, operating in mode (1, 2), these accurate models come along with high complexity compared to other hardware-types and modes of operation. The generic model for mode of operation (2, 1) has the same number of inputs, but a less complex model than the one of MoO (1, 2) and, moreover, is free of singularities. Let us further analyze the differences between these models by bringing a (1, 2) model to a form similar to a (2, 1) model:

Place the origin O_{R} of the robot-fixed reference frame $\Sigma_{\mathrm{R}} = \{O_{\mathrm{R}} : \mathbb{X}_1, \mathbb{X}_2\}$ in the contact point of wheel i. Let the $\mathbb{X}_1$ axis point towards the contact point of wheel j. Then, $l_i = \alpha_i = \alpha_j = 0$. With this choice, the (1, 2) posture kinematic model is given by:

$$\dot{\boldsymbol{x}} = \mathbf{B}(\boldsymbol{x})\boldsymbol{u} \tag{27}$$

$$= \begin{pmatrix} l_j\left(\cos\left(\theta+\beta_i+\beta_j\right)-\cos\left(\theta+\beta_i-\beta_j\right)\right) & 0 & 0 \\ l_j\left(\sin\left(\theta+\beta_i+\beta_j\right)-\sin\left(\theta+\beta_i-\beta_j\right)\right) & 0 & 0 \\ 2\sin\left(\beta_i-\beta_j\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \zeta_1 \\ \zeta_2 \end{pmatrix} \tag{28}$$

with l_j the distance between wheel i and wheel j. The input transformation

$$\eta_v = 2 l_j \sin\left(\beta_j\right) \eta_1 \tag{29}$$

brings the system into the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\beta}_i \\ \dot{\beta}_j \end{pmatrix} = \begin{pmatrix} -\sin(\theta+\beta_i) & 0 & 0 \\ \cos(\theta+\beta_i) & 0 & 0 \\ \dfrac{\sin\left(\beta_i-\beta_j\right)}{l_j \sin\beta_j} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_v \\ \zeta_1 \\ \zeta_2 \end{pmatrix}. \tag{30}$$

In this representation, the only expression depending on β_j is the one corresponding to $\dot{\theta}$. Taking

$$\eta_\theta = \frac{\sin\left(\beta_i-\beta_j\right)}{l_j \sin\beta_j} \eta_v \tag{31}$$

as new input gives

$$\dot{\theta} = \eta_\theta . \tag{32}$$

Equation (31) is a transformation that decouples the equation for $\dot{\theta}$ from the other equations. However, β_j needs to be controlled such that Eq. (31) is satisfied. Solving for β_j gives

$$\beta_j = \arctan\left(\frac{\sin(\beta_i)\,\eta_v}{l_j \eta_\theta + \cos(\beta_i)\,\eta_v}\right) + k\pi \tag{33}$$

with $k \in \mathbb{Z}$ chosen such that β_j is continuous. If a controller was able to make this equation invariant, then the posture kinematic model would simplify to

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\beta}_i \end{pmatrix} = \begin{pmatrix} -\sin(\theta+\beta_i) & 0 & 0 \\ \cos(\theta+\beta_i) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_v \\ \eta_\theta \\ \zeta_1 \end{pmatrix}, \tag{34}$$

which is similar to the model for MoO (2, 1).

The reason why a (2, 1) model is not as accurate as a (1, 2) model for a robot operating in mode (1, 2) is found in Table 1: for models in class (3, 0), (2, 1) and (2, 0) no steering-angle-related vector field is involved in spanning TG. A steering-angle-related vector field can only be involved in spanning TG via Lie-bracket motions. As afore mentioned, Lie-bracket motions can be considered to be slower than pure vector field motions. For (1, 2) and (1, 1) models, such Lie bracket motions are required for spanning TG, that is, for reaching any pose in the plane.

An intuitive interpretation of these arguments is found by considering actuation: in a (2, 1) model, there is no state corresponding to β_j. It is thus not possible to introduce a model of the steering actuator into the model. Making Eq. (33) invariant would require a steering actuator that takes no time for reorientation of wheel j. If such an actuator would be available, then there would be no restriction of mobility due to the sliding constraints at all. This shows that a (2, 1) model does not capture the possible delay that originates from reorientation of wheels. However, such reorientations are necessary for the robot to reach any pose in the plane.

One could argue that also for a (1, 2) model there are unmodeled steering actuators. If a robot has more than two center-steered wheels, then only two of them appear in the model. However, since the ICR is 2-DoF, at most two steering angles can be restricting the motion of the robot at a time. This can be accounted for using a switching strategy (Betourne et al., 1996; Bak et al., 2003) or a clever choice for a coordinate chart of the steering angle manifold M (Thuilot et al., 1996).

6 Singularities

By singularities we mean configurations, at which sliding constraints become linear dependent. Note that this definition is not bound to a rank loss of $\mathbf{C}_1^*$, like for example the definition in Gracia and Tornero (2007). The reason for that is found in the models of class (1, 2). Although a robot that is described by such a model might have more than two steered wheels, only two steering angles appear in the state vector. Linear dependence of the corresponding sliding constraints causes a situation similar to the singularity in the two-wheel case, illustrated in Example 1. We therefore also call it singularity in the multi-wheel case.

Singularities of type A are those, where the wheel axes of all wheels of a robot with coaligned center-steerable wheels (HW-type Vb) coincide. This singularity was introduced in Example 1 as a model-switch, leading to a hybrid WMR model. From a mathematical point of view, for hybrid systems, the meaning of *solution* and its existence and uniqueness are not per se defined (Filippov, 1988). In this special case, however, the meaning of *solution* is rather clear and indeed equal to the meaning of *solution* in the continuous case. From a physical point of view also existence and uniqueness of solutions are given, as a real robot indeed executes a concrete motion in an experiment. The latter could however just be the result of dynamical effects that are not contained in our kinematic model.

The mathematical theory that allows to conclude about the properties of existence and uniqueness is the theory of Carathèodory differential equations and Lebesgue integration. A generalization of the notion of *solution* is called a solution in an extended sense $\boldsymbol{x}(t)$. Such a solution is absolutely continuous and satisfies the differential equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t)$ almost everywhere. For existence, it is required that the right hand side $\boldsymbol{f}(\boldsymbol{x}, t)$ of the equation satisfies the following conditions:

1. $\boldsymbol{f}(\boldsymbol{x}, t)$ must be defined and continuous in x for almost all t;
2. $\boldsymbol{f}(\boldsymbol{x}, t)$ must be measurable in t for each x;
3. there must exist a function $m(t) \geq |\boldsymbol{f}(\boldsymbol{x}, t)|$ that must be summable on every finite interval $\mathcal{I} \ni t$.

Uniqueness is given if there exists a summable function $\boldsymbol{l}(t)$ such that $\boldsymbol{f}(\boldsymbol{x}, t)$ satisfies the following Lipschitz-like condition:

$$||\boldsymbol{f}(\boldsymbol{x}_1, t) - \boldsymbol{f}(\boldsymbol{x}_2, t)|| \leq \boldsymbol{l}(t)||\boldsymbol{x}_1 - \boldsymbol{x}_2||. \quad (35)$$

For a type Vb robot operating in mode (1, 2), the vector field $\boldsymbol{f}(\boldsymbol{x}, t)$ is defined by Eq. (15) with $\boldsymbol{u}$ to be set by a controller. Since the set of configurations Q_2 where the mode switches to (2, 0) is nowhere dense in the configuration manifold, the robot operates in mode (1, 2) for almost all t if this controller does not contain any feedback and is fully actuated also in MoO (2, 0). In such a case, it is not required to consider the (2, 0) model at all. The vector field $\boldsymbol{f}(\boldsymbol{x}, t)$ corresponding to mode (1, 2) indeed satisfies the conditions for existence and uniqueness of a solution in the extended sense. Type A singularities can therefore be ignored in feedforward control, at least from an idealized point of view also ignoring dynamical effects.

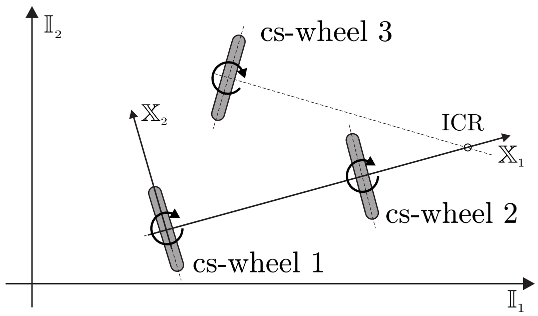

Figure 4. An example for a singularity of type B for a robot with three cs-wheels that are not coaligned: no unique ICR is defined by wheels one and two, but one and three.

In the feedback-case, the situation is more challenging: suppose a reference trajectory is designed in a way that requires the robot to stay in a singular configuration for longer time. In the feedforward-case this does not compromise the Carathèodory conditions, as the robot will still be in $Q \setminus Q_2$ almost always. However, if a feedback controller is active, the robot is forced to enter Q_2 again and again (by the controller), leading to high-frequency switching. Continuity of $\boldsymbol{f}(\boldsymbol{x}, t)$ in x for almost all t is therefore questionable, as is the use of a controller that is based on a (1, 2)-model in such a case. In practical experiments with controllers based solely on (1, 2)-models, a chattering of the steering actuators is observed on such a "singular" trajectory (Gruber and Hofbaur, 2014). The reason for this chattering is that a (1, 2)-controller uses steering actuators to move the ICR and correct small orientation errors of the robot. Since the wheel axes are parallel in a type-A singularity, this problem is ill conditioned. The proper actuators for this task would be the wheel speeds, cf. Example 1.

A singularity of type A is only present for robots with coaligned wheel contact points (HW-type Vb). When the wheel contact points are not on a straight line (HW-type Va), then one or more (but not all) wheel axes may be coaligned. Such a situation is a singularity of type B, shown in Fig. 4. Since only two steering angles appear in the posture kinematic model of the robot, singularities of type B appear similar to singularities of type A if the corresponding wheels are coaligned. The actual difference is that in type B singularities, the number of immediately accessible DoF does not increase because there is at least one more wheel that still restricts motion. Type B singularities can therefore be avoided

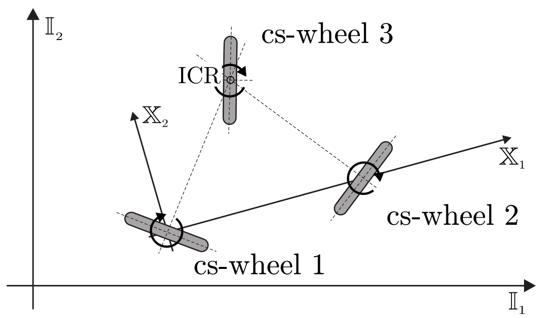

Figure 5. An example for a singularity of type C for a robot with three cs-wheels that are not coaligned: the ICR, defined by wheels one and two, coincides with the contact point of wheel three.

by using a hybrid system model (Bak et al., 2003) or a clever choice of coordinates for M.

A robot is in a type C singularity when the ICR coincides with the contact point of a center-steerable standard wheel, see Fig. 5. The challenges associated with type C singularities are two-fold: on one hand, once the ICR lies exactly in the contact point of a center-steerable wheel then its steering angle cannot be computed from the ICR-relation. The practically more relevant problem is that steering speeds get high when the ICR is required to get close to a wheel contact point. Some works exist that deal with this singularity problem and avoidance strategies (D'Andrea-Novel et al., 1995; Connette et al., 2009; Thuilot et al., 1996; Dietrich et al., 2011).

7 The geometry of wheeled mobile robots

The previous sections tried to require a minimum of preknowledge in differential geometry to make this article easily accessible to a broader audience. A crash-course for control engineers is found in Schlacher and Schöberl (2014). An early, comprehensive treatment of nonholonomy in dynamic systems is found in Neimark and Fufaev (1972). In this section, we analyze wheeled mobile robots from the point of view of differential geometry. This allows us to precisely reason about state manifolds, singularities and conditions for non-degeneracy.

Let us again start at the root cause for restriction of motion, the sliding constraints from Eq. (3). For each standard wheel i, the sliding constraint is defined via an analytic 1-form $\omega_i(\boldsymbol{q})$. Such a constraint reduces the number of degrees of freedom of a robot only if it is integrable, that is, holonomic. The Frobenius theorem provides a criterion to determine if a set of Pfaffian equations is holonomic. Let $\wedge$ be the exterior (wedge) product on forms and define

$$\Omega(\boldsymbol{q}) = \omega_1(\boldsymbol{q}) \wedge \cdots \wedge \omega_{N_\mathrm{c}}(\boldsymbol{q}) \quad (36)$$

as an N_c-form (Chen et al., 1997). A set of constraints is holonomic, if at every point $\boldsymbol{q}$, for all $r = 1, \ldots, N_\mathrm{c}$,

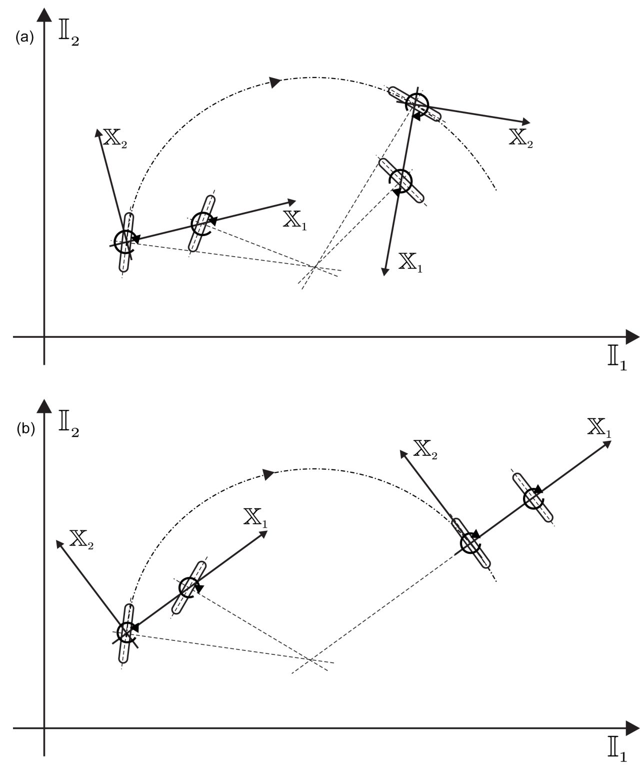

Figure 6. **(a)** Circular motion of a HW-type Vb WMR with constant steering angles. **(b)** Circular motion of a HW-type Vb WMR with constant orientation.

$$F_r(\boldsymbol{q}) = (\mathrm{d}\omega_r \wedge \Omega)(\boldsymbol{q}) = 0, \quad (37)$$

where $\mathrm{d}\omega_r$ denotes the exterior derivative of ω_r. However, we do not just want to reason whether all constraints are integrable or not, but also consider partial integrability. To do so, let us analyze the space spanned by the 1-forms ω in more detail: the collection of 1-forms ω locally spans a linear subspace of the cotangent space

$$D(\boldsymbol{q}) = \mathrm{span}\left\{\omega_1(\boldsymbol{q}), \ldots, \omega_{N_\mathrm{c}}(\boldsymbol{q})\right\} \subset T^*_{\boldsymbol{q}} Q. \quad (38)$$

A family of such subspaces is called a *generalized codistribution* (Cortés et al., 2001). If the dimension of this codistribution is constant on an open neighborhood of $\boldsymbol{q}_0$, then $\boldsymbol{q}_0$ is called a regular point of D. Otherwise, it is a singular point. If the dimension of D is constant on all of Q, then D is called a regular codistribution. Given a generalized codistribution D, let us define its coannihilator D^0 as the *generalized distribution* Δ (in fact, this is exactly the object we were dealing with in Sect. 4).

We are looking for the "largest" connected submanifold X of Q on which the dimension of Δ is constant. On this submanifold, also δ_m – which is the dimension of Δ – and δ_s, together defining the mode of operation, are constant. Given the generalized codistribution D, let $\mathrm{inv}(\Delta)$ be the involutive closure of Δ. Since $\mathrm{inv}(\Delta)$ is involutive and spanned by a locally finitely generated (see Table 1) set of vector fields, it has the maximal integral manifolds property (Isidori, 1995, Theorem 2.1.5). This guarantees the existence of such manifolds X. Now let us introduce these formally:

The submanifold X of Q is an integral manifold of inv(Δ) if $T_x X$ is spanned by inv(Δ)($\boldsymbol{x}$) at each $\boldsymbol{x} \in X$. An open submanifold X of Q is a maximal integral manifold (leaf) if every connected integral manifold of inv(Δ) which intersects X is an open submanifold of X. These leaves thereby are the "largest" connected subsets in Q, on which δ_m and δ_s are constant. A distribution has the maximal integral manifolds property, if there exists a leaf passing through every point on the manifold. Since inv(Δ) has this property, there exists a leaf passing through every point $\boldsymbol{q} \in Q$. These submanifolds X therefore perfectly qualify as state manifolds for the associated control systems.

Since inv(Δ) has the maximal integral manifolds property but may be singular, WMR are simple examples for an interesting class of systems: depending on the initial state, one may obtain different control systems that evolve on manifolds of different dimensions, cf. (Isidori, 1995, Remark 2.2.4). Bullo and Lewis (2005, Chapter 4.5.2) find "work to be done" in the field of singular codistributions.

Let us now apply this theory to wheeled mobile robots. Consider a robot with only one center-steerable or fixed standard wheel. In this case, there is only one 1-form $\omega_1(\boldsymbol{q})$ due to the single sliding constraint. In this special case, the integrability conditions from Frobenius theorem, Eq. (37), are violated globally. The codistribution D is regular, and, as a result, δ_m and δ_s are constant on all of Q. This allows to draw the following conclusions: WMR with only one center-steerable wheel or one fixed standard wheel are (i) nonholonomic, (ii) globally controllable, since the LARC (Isidori, 1995) holds globally, and (iii) the state manifold X of the associated control system is the whole configuration manifold Q.

However, for two sliding constraints, the situation already gets more complicated. Consider the robot with two center-steerable wheels from Example 1. Without loss of generality, place the origin of the robot fixed frame in the contact point of wheel 1 and let the $\mathbb{X}_1$ axis point towards the contact point of wheel 2 (see Fig. 2a). This specific choice of coordinates is just to simplify expressions ($\alpha_1 = 0$, $\alpha_2 = 0$, $l_1 = 0$): the 1-forms are

$$\omega_1 = \cos(\beta_1 + \theta)\mathrm{d}x + \sin(\beta_1 + \theta)\mathrm{d}y$$
$$\omega_2 = \cos(\beta_2 + \theta)\mathrm{d}x + \sin(\beta_2 + \theta)\mathrm{d}y + l_2 \sin(\beta_2)\mathrm{d}\theta$$

and Ω evaluates to:

$$\begin{aligned}\Omega = & -\sin(\beta_1 - \beta_2)\mathrm{d}x \wedge \mathrm{d}y \\ & + \cos(\beta_1 + \theta) l_2 \sin(\beta_2)\mathrm{d}x \wedge \mathrm{d}\theta \\ & + \sin(\beta_1 + \theta) l_2 \sin(\beta_2)\mathrm{d}y \wedge \mathrm{d}\theta. \end{aligned} \quad (39)$$

For the integrability conditions one obtains

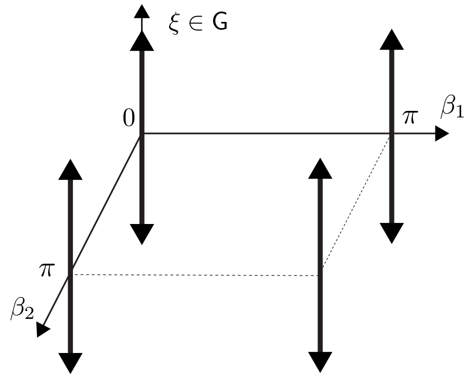

Figure 7. For two sliding constraints, the involutive closure of the coannihilator of the generalized codistribution D foliates Q into five separate leaves. The four leaves indicated by the bold rays correspond to the leaves at the four singular points of D. The fifth leaf fills the remaining open space (schematic).

$$F_1 = l_2 \sin(\beta_2)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\theta \wedge \mathrm{d}\beta_2 = 0 \quad (40)$$
$$F_2 = l_2 \sin(\beta_1)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}\theta \wedge \mathrm{d}\beta_1 = 0, \quad (41)$$

which both hold only when the axes of both wheels coincide, that is, both conditions hold on

$$Q_2 = \{(x, y, \theta, \beta_1, \beta_2) \,|\, \beta_1 \in \{0, \pi\} \text{ and } \beta_2 = \{0, \pi\}\} \quad (42)$$

which contains four disjoint submanifolds, each of them corresponding to a connected set of singular points of D. This generalized codistribution D foliates Q into five separate leaves, see Fig. 7. At every regular point $\boldsymbol{q} \in Q_1 = Q \setminus Q_2$, the dimension of inv($\Delta$) is equal to the dimension of Q, that is, 5. On Q_2, ω_1 and ω_2 become linear dependent and take the form

$$\omega = \cos(\theta)\mathrm{d}x + \sin(\theta)\mathrm{d}y \quad (43)$$

and the dimension of inv(Δ) is reduced to 3. These singular points of the codistribution D are those that were introduced as *singularities of type A* (cf. Table 2). An example for a robot with two steered wheels in a type-A singularity is shown in Fig. 2b. Summing up, this gives the following results: WMR with two center-steerable or fixed standard wheels are (i) nonholonomic, since sliding constraints are at most partly integrable; (ii) the state manifold X of the currently active model is the leaf determined by the steering angle configuration; and (iii) globally controllable, since the LARC holds at each $\boldsymbol{x} \in X$. Note, however, that both, the dimension of X and the rank of the Lie algebra are not constant on all of Q.

For three sliding constraints, the integrability condition (Eq. 37) holds globally. In similar coordinates as in the case of two sliding constraints, the condition $\Omega = 0$ from Eq. (36) can be solved for β_3. Thereby, one obtains

$$\beta_3 = -\arctan\left(\frac{-l_2\cos(\alpha_3-\beta_1+\beta_2)+l_2\cos(\alpha_3-\beta_1-\beta_2)}{l_2\sin(\alpha_3-\beta_1+\beta_2)-l_2\sin(\alpha_3-\beta_1-\beta_2)+2l_3\sin(\beta_1-\beta_2)}\right) + k\pi \quad (44)$$

with $k \in \mathbb{Z}$ chosen such that β_3 is continuous. This condition ensures the existence of a unique ICR. On the surface N defined by this equation, two of the three 1-forms (sliding constraints) are linear dependent and similar conclusions as for the two-sliding-constraint-case can be drawn.

Constraint (Eq. 44) is singular on $\beta_1, \beta_2 \in Q_2$, in which case no unique ICR is defined by wheel 1 and 2. This is the singularity introduced in Sect. 6 as *singularity of type B*. On this singularity, 1-forms ω_1 and ω_2 become linear dependent. However, if $\alpha_3 \neq k_3\pi$, $k_3 \in \mathbb{Z}$ (wheels $1-3$ are not colinear), then ω_1 and ω_3 are independent. The ICR is therefore defined by the axes of wheel 1 and wheel 3. An example for such a singularity is shown in Fig. 4. To avoid singularities of type B, it is required to either switch (Betourne et al., 1996) to coordinates β_1, β_3 or β_2, β_3 or choose a completely different chart on N, see Thuilot et al. (1996) for a clever choice. Note that singularities of type B are no singularities of the codistribution D.

A *singularity of type C* appears when only the denominator of the fraction in the arctangent in Eq. (44) is zero. This is the case when the ICR is placed in the contact point of wheel 3. An example for such a singularity is shown in Fig. 5.

If $\alpha_3 = k_3\pi$, $k_3 \in \mathbb{Z}$ and $\boldsymbol{q} \in Q_3$,

$$Q_3 = \{(x, y, \theta, \beta_1, \beta_2, \beta_3) \,|\, \beta_1 \in \{0, \pi\},\ \beta_2 = \{0, \pi\}, \beta_3 = \{0, \pi\}\}, \quad (45)$$

then all 1-forms become linear dependent and the dimension of D reduces to 1 at these singular points. This is the case when all wheel axes are co-aligned and reduces to the same situation as for co-aligned wheels in the two-wheel case. Consistently, these singularities are also classified as *singularities of type A*.

The foliation of Q for three sliding constraints is illustrated in Fig. 8. As in the two-wheel case, inv(Δ) foliates Q into several structurally different leaves: (i) the dashed horizontal planes are the leaves that correspond to a robot in rest ($\xi = \text{const.}$). Only a nowhere dense subset of steering angle configurations allow motion. One of those is (ii) the dash-dotted surface defined by Eq. (44). Another one is the dash-dotted plane $\beta_1, \beta_2 = 0$. Only if $\alpha_3 = k_3\pi$, $k_3 \in \mathbb{Z}$, then also the (iv) bold vertical arrows are leaves, corresponding to the case when all wheel axes are co-aligned.

It can be concluded that WMR with three center-steerable wheels are (i) nonholonomic, (ii) the state manifold X of the currently active model is the leaf determined by the steering angle configuration; and (iii) globally controllable, since the LARC holds at each $\boldsymbol{x} \in X$.

The case of three sliding constraints can be analogously extended to an arbitrary number of wheels: for every further wheel, either its axis must be co-aligned to at least one of the existing wheels or coordinated by a coordination function asserting the existence of a unique ICR.

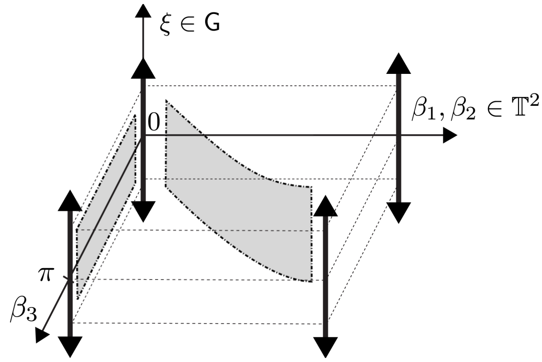

Figure 8. For three sliding constraints, the involutive closure of the coannihilator of the generalized codistribution D foliates Q into several leaves (schematic).

8 Conclusions

This article presented a detailed view on the kinematics of single-bodied wheeled mobile robots.

One suggestion is to drop the notion of degeneracy of wheeled mobile robots and replace it with controllability. The usage of this term is consistent with the standard definition in control theory. Furthermore, a simple condition for controllability was presented.

Based on this controllability study, a classification of wheeled mobile robots into six hardware types is introduced. This classification is solely based on the type, location and number of wheels. Each hardware type is able to operate in one or more modes of operation. A mode of operation corresponds to a set of configurations, in which a specific model is an accurate representation of the kinematic capabilities of the robot.

Moreover, we provide a detailed analysis of the geometry of wheeled mobile robots by which we are able to give a general view on state manifolds and singularities.

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