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Supplement of

A predefined-time radial basis function (RBF) neural network tracking control method considering actuator faults for a new type of spraying robot

Jingang Zhao et al.

Correspondence to: Yiwen Li (lee_yiwen@163.com)

Supplementary Information

S1 Modeling process of n rotary joint robotic arm

The Lagrangian equation[20] for the attitude adjustment mechanism is given by:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \boldsymbol{Q_i} \quad (i = 1,2) \tag{1}$$

where $\boldsymbol{L}(\boldsymbol{q_i}, \dot{\boldsymbol{q}}_i) = \boldsymbol{T} - \boldsymbol{U}$ is the difference between the kinetic energy $\boldsymbol{T}$ and the potential energy $\boldsymbol{U}$ of the attitude mechanism, the $\boldsymbol{q_i}$ is the joint variable, $\dot{\boldsymbol{q}}_i$ is the its angular velocity, $\boldsymbol{Q_i}$ is the generalized force corresponding to the generalized coordinates.

The kinetic energy, potential energy and generalized force of the attitude adjustment mechanism is given by:

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T M \dot{\boldsymbol{q}} \tag{2}$$

$$\boldsymbol{U} = \sum_{i=1}^{2} m_i \, \boldsymbol{g^T} \cdot^{\boldsymbol{0}} \boldsymbol{p_{ci}} \tag{3}$$

$$\boldsymbol{Q} = \boldsymbol{\tau} + \boldsymbol{J^T F} \tag{4}$$

where$\boldsymbol{M} = \sum_{i=1}^{2}\left(m_i \boldsymbol{J_L^{(i)T}} \boldsymbol{J_L^{(i)}} + \boldsymbol{J_A^{(i)T}} \boldsymbol{I_i} \boldsymbol{J_A^{(i)}}\right)$, g is the gravitational acceleration vector in the base coordinate system, $\boldsymbol{0p_{ci}}$ is the center of mass vector from the coordinate origin to link i in the base coordinate system. $\tau$and$F$denote the joint force vector and the contact force vector between the hand end and the environment, respectively.

Substituting the kinetic energy, potential energy and generalized force of the attitude mechanism into the Lagrangian equation (3), the robot dynamics equations can be obtained as:

$$\sum_{j=1}^{2} \boldsymbol{M_{ij}} \ddot{\boldsymbol{q}}_j + \sum_{j=1}^{2}\sum_{k=1}^{2} h_{ijk} \, \dot{\boldsymbol{q}}_{\boldsymbol{j}} \dot{\boldsymbol{q}}_{\boldsymbol{k}} + \boldsymbol{G}_i = \boldsymbol{Q_i} \tag{5}$$

where $\boldsymbol{G_i} = \sum_{j=1}^{2} m_j \, \boldsymbol{g^T J_{Li}^{(j)}}$, $h_{ijk} = \frac{\partial \boldsymbol{M_{ij}}}{\partial \boldsymbol{q_k}} - \frac{1}{2}\frac{\partial \boldsymbol{M_{jk}}}{\partial \boldsymbol{q_i}}$, $i = 1,2 j = 1,2$.

In practice, due to the difference between the mathematical model and the actual model, the model function of the robot can be expressed as follows:

$$\begin{cases} \boldsymbol{M(q)} = \boldsymbol{M_0(q)} + \boldsymbol{\Delta M(q)} \\ \boldsymbol{C(q,\dot{q})} = \boldsymbol{C_0(q,\dot{q})} + \boldsymbol{\Delta C(q,\dot{q})} \\ \boldsymbol{G(q)} = \boldsymbol{G_0(q)} + \boldsymbol{\Delta G(q)} \end{cases} \tag{6}$$

where$\boldsymbol{M_0(q)}, \boldsymbol{C_0(q,\dot{q})}$ and $\boldsymbol{G_0(q)}$ is nominal values $\boldsymbol{\Delta M(q)}, \boldsymbol{\Delta C(q,\dot{q})}, \boldsymbol{\Delta G(q)}$ is modeling matrix error.

S2 The principle of fitting unknown function f with S2 RBF neural network

In (12), f is an unknown nonlinear function, and it is difficult to ensure the realization of the control law u. RBF network is one kind of neural network, which belongs to the forward type network and consists of 3-layer network, and its network structure is shown in Fig. 5:

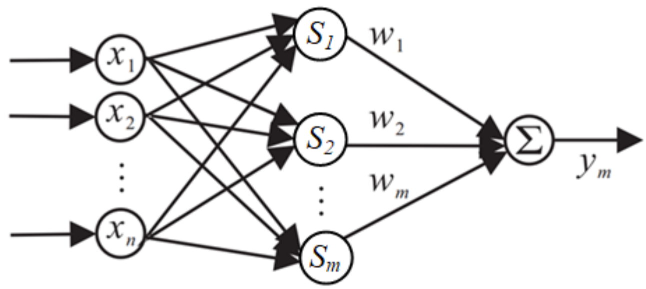

Figure S1 RBF neural network structure diagram

Layer 1 $x = [x_1, ..., x_n]^T$ is the input layer, which is the input to the network, and n is the dimension of the input.

Layer 2 is the hidden layer, whose output is $\boldsymbol{S}(\boldsymbol{x}) = [\boldsymbol{S_1}, \ldots, \boldsymbol{S_m}]^{\boldsymbol{T}}$, uses a Gaussian basis function as the affiliation function of the input layer which is:

$$\boldsymbol{S_j}(\boldsymbol{x}) = exp(-\frac{\|\boldsymbol{x}-\boldsymbol{c}\|^2}{2\boldsymbol{b}^2}) \tag{7}$$

where c is the coordinate vector of the center point of the Gaussian basis function of the implied layer. b is the width of the Gaussian basis function of the implied layer. And $j = 1,2, \ldots, m$.

The weights of the RBF neural network are:

$$\boldsymbol{W} = [\boldsymbol{W_1}, \ldots, \boldsymbol{W_m}]^T \tag{8}$$

Layer 3 is the output layer, and the output of the neural network is:

$$y = \boldsymbol{W^T S}(\boldsymbol{x}) = \boldsymbol{W_1 S_1} + \cdots + \boldsymbol{W_m S_m} \tag{9}$$

RBF network is utilized to approximate f_i. The input to the network is taken as $x = [e, \dot{e}, \ddot{e}]^T$, and the output is:

$$\boldsymbol{f_i} = \boldsymbol{W_i^T S_i}(\boldsymbol{x}) + \boldsymbol{\phi_i} \tag{10}$$

where ϕ_i is the approximation error of the network.

$$\boldsymbol{f} = \boldsymbol{W^T S}(\boldsymbol{x}) + \boldsymbol{\phi} \tag{11}$$

S3 PRC controller design

This article uses RBF neural network to fit the unknown function f, and its principle is described in **S2** of Supporting Information.

The multi-joint robot dynamics model of (12) is used:

$$\ddot{\boldsymbol{q}} = \boldsymbol{f}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q})\boldsymbol{u} + \boldsymbol{g}(\boldsymbol{q})\boldsymbol{D}(\boldsymbol{t})$$

Defining $p(t) = \frac{e(t)}{\rho(t)}$, $\boldsymbol{x}_1 = \boldsymbol{q}$, $\boldsymbol{x}_2 = \dot{\boldsymbol{q}}$, system (12) can be rewritten as:

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2 \\ \dot{\boldsymbol{x}}_2 = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{t}) + \boldsymbol{g}(\boldsymbol{x}_1)\boldsymbol{v}(\boldsymbol{u}) + \boldsymbol{g}(\boldsymbol{x}_1)\boldsymbol{D}(\boldsymbol{t}) \\ \boldsymbol{y} = \boldsymbol{x}_1 \\ \boldsymbol{f}(\boldsymbol{q},\dot{\boldsymbol{q}}) = -\boldsymbol{M}^{-1}(\boldsymbol{q})[\boldsymbol{V}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q})] \\ \boldsymbol{M}^{-1}(\boldsymbol{q}) = \boldsymbol{g}(\boldsymbol{q}) \end{cases} \tag{12}$$

Defining the error transformation function as (22):

$$\boldsymbol{z}(\boldsymbol{p}) = \frac{\boldsymbol{p}}{\boldsymbol{1} - \boldsymbol{p}^2}$$

where $\boldsymbol{p} = \boldsymbol{p}(t)$.

Assume that $\boldsymbol{y}_d$ is the desired angle and $\boldsymbol{y}_d$ has a second order derivative.

Defining the tracking error as:

$$\boldsymbol{e}_1 = \boldsymbol{y} - \boldsymbol{y}_d \tag{13}$$

The derivation of e yields:

$$\dot{\boldsymbol{e}}_1 = \dot{\boldsymbol{y}} - \dot{\boldsymbol{y}}_d = \dot{\boldsymbol{\rho}}(\boldsymbol{t})\boldsymbol{p}_1 + \dot{\boldsymbol{z}}_1\boldsymbol{\rho}(t)\frac{\partial \boldsymbol{p}_1}{\partial \boldsymbol{z}_1} \tag{14}$$

Defining the angular velocity tracking error as:

$$\boldsymbol{e}_2 = \boldsymbol{x}_2 - \boldsymbol{x}_{2d} = \boldsymbol{x}_2 - \dot{\boldsymbol{y}}_d \tag{15}$$

Substituting $\dot{y} = \dot{x}_1 = x_2$ gives the result by shifting the terms:

$$\dot{\boldsymbol{z}}_1 = \frac{\boldsymbol{x}_2 - \dot{\boldsymbol{y}}_d - \dot{\boldsymbol{\rho}}(t)\boldsymbol{p}_1}{\boldsymbol{\rho}(t)\frac{\partial \boldsymbol{p}_1}{\partial \boldsymbol{z}_1}} = \boldsymbol{\varUpsilon}_1 + \boldsymbol{E}_1(\boldsymbol{x}_2 - \dot{\boldsymbol{y}}_d) \tag{16}$$

where $E_1 = \frac{1}{\rho(t)\frac{\partial p_1}{\partial z_1}}$, $\varUpsilon_1 = -\dot{\rho}(t)p_1E_1$.

Substituting $\boldsymbol{x}_2 = \boldsymbol{e}_2 + \dot{\boldsymbol{y}}_d$ gives:

$$\dot{\boldsymbol{z}}_1 = \boldsymbol{\varUpsilon}_1 + \boldsymbol{E}_1(\boldsymbol{x}_2 - \dot{\boldsymbol{y}}_d) \tag{17}$$

The derivation of e_2 is obtained:

$$\dot{\boldsymbol{e}}_2 = \dot{\boldsymbol{x}}_2 - \ddot{\boldsymbol{y}}_d = \dot{\boldsymbol{\rho}}(t)\boldsymbol{p}_2 + \dot{\boldsymbol{z}}_2\boldsymbol{\rho}(t)\frac{\partial \boldsymbol{p}_2}{\partial \boldsymbol{z}_2} \tag{18}$$

With (43)，it will be rewritten as:

$$\boldsymbol{g}(x_1)\boldsymbol{v}(u) - \boldsymbol{g}(x_1)\boldsymbol{d}(t) + \boldsymbol{f}(x,t) - \ddot{\boldsymbol{y}}_d = \dot{\boldsymbol{\rho}}(t)\boldsymbol{p}_2 + \dot{\boldsymbol{z}}_2\boldsymbol{\rho}(t)\frac{\partial \boldsymbol{p}_2}{\partial \boldsymbol{z}_2}$$

Shifting the terms gives:

$$\dot{\boldsymbol{z}}_2 = \frac{1}{\boldsymbol{\rho}(t)\frac{\partial \boldsymbol{p}_2}{\partial \boldsymbol{z}_2}} \cdot (\boldsymbol{g}(\boldsymbol{x}_1)\boldsymbol{v}(\boldsymbol{u}) + \boldsymbol{g}(\boldsymbol{x}_1)\boldsymbol{D}(t) + \boldsymbol{f}(\boldsymbol{x}, t) - \ddot{\boldsymbol{y}}_d - \dot{\boldsymbol{\rho}}(t)\boldsymbol{p}_2)$$

$$= \boldsymbol{\Upsilon}_2 + \boldsymbol{E}_2(\boldsymbol{g}(\boldsymbol{x}_1)\boldsymbol{v}(\boldsymbol{u}) + \boldsymbol{g}(\boldsymbol{x}_1)\boldsymbol{D}(t) + \boldsymbol{f}(\boldsymbol{x}, t) - \ddot{\boldsymbol{y}}_d) \tag{19}$$

where $\boldsymbol{E}_2 = \frac{1}{\boldsymbol{\rho}(t)\frac{\partial \boldsymbol{p}_2}{\partial \boldsymbol{z}_2}}$, $\boldsymbol{\Upsilon}_2 = -\dot{\boldsymbol{\rho}}(t)\boldsymbol{p}_2\boldsymbol{E}_2$.

From Eq. (32), $\frac{\partial \boldsymbol{p}}{\partial \boldsymbol{z}} = \frac{(1-\boldsymbol{p}^2)^2}{1+\boldsymbol{p}^2} \leq 1$ holds when $-1 < \boldsymbol{p} < 1$. It is obtained by combining **Assumption 2**:

$$\boldsymbol{E}_1\boldsymbol{g} = \frac{\boldsymbol{g}}{\boldsymbol{\rho}(t)\frac{\partial \boldsymbol{p}_1}{\partial \boldsymbol{z}_1}} \geq \frac{\boldsymbol{g}}{\boldsymbol{\rho}(0)} = \widetilde{\boldsymbol{g}} > 0 \tag{20}$$

Take the Lyapunov function as:

$$V_1 = \frac{1}{2}\boldsymbol{z}_1^T\boldsymbol{z}_1 + \frac{1}{2\mu_1}\widetilde{\boldsymbol{g}}\widetilde{\boldsymbol{\ell}}_1^T\widetilde{\boldsymbol{\ell}}_1 \tag{21}$$

where $\widetilde{\boldsymbol{\ell}}_1 = \boldsymbol{\ell}_1 - \widehat{\boldsymbol{\ell}}_1$, $\widehat{\boldsymbol{\ell}}_1$ denotes the estimated value of $\boldsymbol{\ell}_1$, $\widetilde{\boldsymbol{\ell}}_1$ denotes the estimation error, and $\mu_1 > 0$ denotes the design parameters.

Differentiating V_1 with respect to time yields:

$$\dot{V}_1 = \boldsymbol{z}_1(\boldsymbol{\Upsilon}_1 + \boldsymbol{E}_1\boldsymbol{x}_2 - \boldsymbol{E}_1\dot{\boldsymbol{y}}_d) - \frac{1}{\mu_1}\widetilde{\boldsymbol{g}}\widetilde{\boldsymbol{\ell}}_1\dot{\widehat{\boldsymbol{\ell}}}_1 \tag{22}$$

The nonlinear function F_1 is defined as:

$$\boldsymbol{F}_1 = \boldsymbol{\Upsilon}_1 + \boldsymbol{E}_1\boldsymbol{x}_2 - \boldsymbol{E}_1\dot{\boldsymbol{y}}_d + \frac{1}{2}\boldsymbol{z}_1 \tag{23}$$

The unknown nonlinear function F_1 is estimated using the RBF neural network system, denoted as follows:

$$\boldsymbol{F}_1 = \boldsymbol{W}_1^T\boldsymbol{S}_1(\boldsymbol{x}, t) + \boldsymbol{\phi}_1(\boldsymbol{x}, t) \tag{24}$$

where $\boldsymbol{x} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \dot{\boldsymbol{y}}_d]^T$.

With (39) and (40), (38) can be described as:

$$\dot{V}_1 = \boldsymbol{z}_1 W_1^T\boldsymbol{S}_1(x, t) + \boldsymbol{z}_1\boldsymbol{\phi}_1(\boldsymbol{x}, t) - \frac{1}{\mu_1}\widetilde{\boldsymbol{g}}\widetilde{\boldsymbol{\ell}}_1\dot{\widehat{\boldsymbol{\ell}}}_1 - \frac{1}{2}\boldsymbol{z}_1^T\boldsymbol{z}_1 \tag{25}$$

Using Young's inequality, it can be obtained that:

$$\boldsymbol{z}_1\boldsymbol{W}_1^T\boldsymbol{S}_1 \leq \frac{\boldsymbol{z}_1^T\boldsymbol{S}_1^T\boldsymbol{W}_1^T\boldsymbol{W}_1\boldsymbol{S}_1\boldsymbol{z}_1}{4T} + T \tag{26}$$

$$\boldsymbol{z}_1\boldsymbol{\phi}_1(\boldsymbol{x}, t) \leq \frac{1}{2}\boldsymbol{z}_1^T\boldsymbol{z}_1 + \frac{1}{2}\boldsymbol{\phi}_1^T\boldsymbol{\phi}_1(\boldsymbol{x}, t) \tag{27}$$

where $\tau_h > 0$ is a design parameter.

with

$$\dot{V}_1 \leq \frac{\boldsymbol{z}_1^T\boldsymbol{S}_1^T\boldsymbol{W}_1^T\boldsymbol{W}_1\boldsymbol{S}_1\boldsymbol{z}_1}{4T} + T + \frac{1}{2}\boldsymbol{z}_1^T\boldsymbol{z}_1 + \frac{1}{2}\boldsymbol{\phi}_1^T\boldsymbol{\phi}_1(x, t) - \frac{1}{\mu_1}\widetilde{\boldsymbol{g}}\widetilde{\boldsymbol{\ell}}_1\dot{\widehat{\boldsymbol{\ell}}}_1 - \frac{1}{2}\boldsymbol{z}_1^T\boldsymbol{z}_1$$

$$= \frac{\boldsymbol{z}_1^T\boldsymbol{S}_1^T\boldsymbol{W}_1^T\boldsymbol{W}_1\boldsymbol{S}_1\boldsymbol{z}_1}{4T} + T + \frac{1}{2}\boldsymbol{\phi}_1^T\boldsymbol{\phi}_1(x, t) - \frac{1}{\mu_1}\widetilde{\boldsymbol{g}}\widetilde{\boldsymbol{\ell}}_1\dot{\widehat{\boldsymbol{\ell}}}_1$$

According to **Assumption 1** (ω_1 and ε_1 are unknown), in order to improve the robustness of the system to faults, the upper and lower bounds of the fault parameters are defined as:

$$\boldsymbol{\underline{\omega}_1 = inf(E_1 g_1 \omega)},\ \boldsymbol{\vartheta_1 = \frac{1}{\underline{\omega}_1}},\ \boldsymbol{\xi_1 = sup(E_1 g_1 \varepsilon_1)} \tag{28}$$

The following Lyapunov function is utilized:

$$\bar{V}_1 = V_1 + \frac{1}{2l_1}\underline{\omega}_1 \tilde{\vartheta}_1^T \tilde{\vartheta}_1 + \frac{1}{2r_1}\tilde{\xi}_1^T \tilde{\xi}_1 \tag{29}$$

where $v > 0, r_1 > 0$ are design parameters.$\tilde{\vartheta}_1 = \vartheta_1 - \hat{\vartheta}_1$,$\tilde{\xi}_1 = \xi_1 - \hat{\xi}_1$represent estimation errors. $\hat{\vartheta}_1$and $\hat{\xi}_1$are estimates of $\vartheta_1$and ξ_1, respectively.

Differentiating $\bar{V}_1$ with respect to time yields:

$$\dot{\bar{V}}_1 = \dot{V}_1 - \frac{1}{l_1}\underline{\omega}_1 \tilde{\vartheta}_1 \dot{\hat{\vartheta}}_1 - \frac{1}{r_1}\tilde{\xi}_1 \dot{\hat{\xi}}_1 \tag{30}$$

Defining $\ell_1 = \frac{||W_1||^2}{\bar{h}}$ with:

$$\dot{\bar{V}}_1 \leq \frac{\tilde{\ell}_1 \tilde{g} z_1^T S_1^T S_1 z_1}{4T} + T + \frac{1}{2} z_1^T z_1 + \frac{1}{2}\phi_1^T \phi_1(x,t) - \frac{1}{\mu_1}\tilde{g}\tilde{\ell}_1 \dot{\hat{\ell}}_1 - \frac{1}{l_1}\underline{\omega}_1 \tilde{\vartheta}_1 \dot{\hat{\vartheta}}_1 - \frac{1}{r_1}\tilde{\xi}_1 \dot{\hat{\xi}}_1 \quad \leq \frac{\tilde{\ell}_1 \tilde{g} z_1^T S_1^T S_1 z_1}{4T} + T + \frac{1}{2}\phi_1^T \phi_1(x,t) - \frac{1}{\mu_1}\tilde{g}\tilde{\ell}_1 \dot{\hat{\ell}}_1 - \frac{1}{l_1}\underline{\omega}_1 \tilde{\vartheta}_1 \dot{\hat{\vartheta}}_1 - \frac{1}{r_1}\tilde{\xi}_1 \dot{\hat{\xi}}_1 \tag{31}$$

Designing adaptive laws for:

$$\dot{\hat{\ell}}_1 = \frac{\mu_1 z_1^2 S_1^T S_1}{4T} - \Gamma_1 \hat{\ell}_1 \tag{32}$$

$$\dot{\hat{\xi}}_1 = r_1 z_1 \tanh\left(\frac{z_1}{a_1}\right) - b_1 \hat{\xi}_1 \tag{33}$$

$$\dot{\hat{\vartheta}}_1 = -c_1 \hat{\vartheta}_1 \tag{34}$$

where $a_1 > 0$,$b_1 > 0$,$c_1 > 0$ are design parameters.

Substituting (31)~(34) with (31), it can be obtained:

$$\begin{aligned} \dot{\bar{V}}_1 &\leq T + \frac{1}{2}\phi_1^T \phi_1 + \frac{\Gamma_1 \tilde{g}}{\mu_1}\tilde{\ell}_1 \hat{\ell}_1 + \frac{\underline{\omega}_1 c_1}{l_1}\tilde{\vartheta}_1 \hat{\vartheta}_1 + \xi_1\left[|z_1| - z_1 \tanh\left(\frac{z_1}{a_1}\right)\right] + \frac{b_1}{r_1}\tilde{\xi}_1 \hat{\xi}_1 \\ &\leq T + \frac{1}{2}\phi_1^T \phi_1 + \frac{\Gamma_1 \tilde{g}}{\mu_1}\tilde{\ell}_1 \hat{\ell}_1 + \frac{\underline{\omega}_1 c_1}{l_1}\tilde{\vartheta}_1 \hat{\vartheta}_1 + 0.2785 a_1 \xi_1 + \frac{b_1}{r_1}\tilde{\xi}_1 \hat{\xi}_1 \end{aligned} \tag{35}$$

Using Young's inequality, it can be obtained that:

$$\tilde{\ell}_1 \hat{\ell}_1 = (\ell_1 - \tilde{\ell}_1)\tilde{\ell}_1 \leq -\frac{1}{2}\tilde{\ell}_1^T \tilde{\ell}_1 + \frac{1}{2}\ell_1^T \ell_1 \tag{36}$$

$$\tilde{\vartheta}_1 \hat{\vartheta}_1 = (\vartheta_1 - \tilde{\vartheta}_1)\tilde{\vartheta}_1 \leq -\frac{1}{2}\tilde{\vartheta}_1^T \tilde{\vartheta}_1 + \frac{1}{2}\vartheta_1^T \vartheta_1 \tag{37}$$

$$\tilde{\xi}_1 \hat{\xi}_1 = (\xi_1 - \tilde{\xi}_1)\tilde{\xi}_1 \leq -\frac{1}{2}\tilde{\xi}_1^T \tilde{\xi}_1 + \frac{1}{2}\xi_1^T \xi_1 \tag{38}$$

Substituting (36)~(38) with (35), and using Young's inequality and Lemma 1, (52) can be further derived as:

$$\dot{\bar{V}}_1 \leq -\frac{1}{2\mu_1}\Gamma_1 \tilde{g}\tilde{\ell}_1^T \tilde{\ell}_1 - \frac{1}{2l_1}\underline{\omega}_1 c_1 \tilde{\vartheta}_1^T \tilde{\vartheta}_1 - \frac{1}{2r_1} b_1 \tilde{\xi}_1^T \tilde{\xi}_1 + \frac{1}{2\mu_1}\Gamma_1 \tilde{g}\ell_1^T \ell_1 + \frac{1}{2l_1}\underline{\omega}_1 c_1 \vartheta_1^T \vartheta_1 + \frac{1}{2r_1} b_1 \xi_1^T \xi_1 + T + \frac{1}{2}\phi_1^T \phi_1 + 0.2785 a_1 \xi_1 \tag{39}$$

From Eq. (35), $\frac{\partial p}{\partial z} = \frac{(1-p^2)^2}{1+p^2} \leq 1$ holds when$-1 < p < 1$. It is obtained by combining **Assumption 2**:

$$\boldsymbol{E}_2\boldsymbol{g} = \frac{g}{\rho(t)\frac{\partial \boldsymbol{p_2}}{\partial \boldsymbol{z_2}}} \geq \frac{g}{\rho(0)} = \widetilde{\boldsymbol{g}} > 0 \tag{40}$$

Take the Lyapunov function as:

$$V_2 = \frac{1}{2}\boldsymbol{z}_2^T\boldsymbol{z}_2 + \frac{1}{2\mu_2}\widetilde{\boldsymbol{g}}\boldsymbol{\ell}_2^T\boldsymbol{\ell}_2 \tag{41}$$

Differentiating V_2 with respect to time yields:

$$\dot{V}_2 = \boldsymbol{z_2}[\boldsymbol{\Upsilon_2} + \boldsymbol{E_2}(\boldsymbol{g}(\boldsymbol{x_1})\boldsymbol{v}(\boldsymbol{u}) + \boldsymbol{g}(\boldsymbol{x_1})\boldsymbol{D}(t) + \boldsymbol{f}(\boldsymbol{x},t) - \ddot{\boldsymbol{y}}_{\boldsymbol{d}}] - \frac{1}{\mu_2}\widetilde{\boldsymbol{g}}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{2}}\dot{\hat{\boldsymbol{\ell}}}_{\boldsymbol{2}} \tag{42}$$

The nonlinear function F_2 is defined as:

$$F_2 = \boldsymbol{\Upsilon_2} + \boldsymbol{E_2}[\boldsymbol{g}(\boldsymbol{x_1})\boldsymbol{D}(t) + \boldsymbol{f}(\boldsymbol{x},t) - \ddot{\boldsymbol{y}}_{\boldsymbol{d}}] + \frac{1}{2}\boldsymbol{z_2} \tag{43}$$

The unknown nonlinear function F_2 is estimated using the RBF neural network system, denoted as follows:

$$F_2 = \boldsymbol{W}_{\boldsymbol{2}}^{\boldsymbol{T}}\boldsymbol{S_2}(\boldsymbol{x},t) + \boldsymbol{\phi_2}(\boldsymbol{x},t) \tag{44}$$

where $\boldsymbol{x} = [\boldsymbol{x_1}, \boldsymbol{x_2}, \ddot{\boldsymbol{y}}_{\boldsymbol{d}}]^T$.

Substituting (44) with (42), it is obtained by:

$$\dot{V}_2 = \boldsymbol{z_2}\boldsymbol{W}_{\boldsymbol{2}}^{\boldsymbol{T}}\boldsymbol{S_2}(\boldsymbol{x},t) + \boldsymbol{z_2}\boldsymbol{\phi_2}(\boldsymbol{x},t) + \boldsymbol{z_2}\boldsymbol{E_2}\boldsymbol{g}(\boldsymbol{x_1})\boldsymbol{v}(\boldsymbol{u}) - \frac{1}{\mu_2}\widetilde{\boldsymbol{g}}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{2}}\dot{\hat{\boldsymbol{\ell}}}_{\boldsymbol{2}} - \frac{1}{2}\boldsymbol{z}_2^T\boldsymbol{z}_2 \tag{45}$$

Using Young's inequality, it can be obtained that:

$$\boldsymbol{z_2}\boldsymbol{W}_{\boldsymbol{2}}^{\boldsymbol{T}}\boldsymbol{S_2} \leq \frac{\boldsymbol{z}_2^T\boldsymbol{S}_2^T\boldsymbol{W}_2^T\boldsymbol{W}_2\boldsymbol{S}_2\boldsymbol{z}_2}{4T} + T \tag{46}$$

$$\boldsymbol{z_2}\boldsymbol{\phi_2}(\boldsymbol{x},t) \leq \frac{1}{2}\boldsymbol{z}_2^T\boldsymbol{z}_2 + \frac{1}{2}\boldsymbol{\phi}_2^T\boldsymbol{\phi}_2(x,t) \tag{47}$$

with

$$\begin{aligned}\dot{V}_2 &\leq \frac{\boldsymbol{z}_2^T\boldsymbol{S}_2^T\boldsymbol{W}_2^T\boldsymbol{W}_2\boldsymbol{S}_2\boldsymbol{z}_2}{4T} + T + \boldsymbol{z_2}\boldsymbol{E}_2\boldsymbol{g}(\boldsymbol{x_1})\boldsymbol{v}(\boldsymbol{u}) + \frac{1}{2}\boldsymbol{z}_2^T\boldsymbol{z}_2 + \frac{1}{2}\boldsymbol{\phi}_2^T\boldsymbol{\phi}_2(x,t) - \frac{1}{\mu_2}\widetilde{\boldsymbol{g}}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{2}}\dot{\hat{\boldsymbol{\ell}}}_{\boldsymbol{2}} - \frac{1}{2}\boldsymbol{z}_2^T\boldsymbol{z}_2 \\ &= \frac{\boldsymbol{z}_2^T\boldsymbol{S}_2^T\boldsymbol{W}_2^T\boldsymbol{W}_2\boldsymbol{S}_2\boldsymbol{z}_2}{4T} + T + \boldsymbol{z_2}\boldsymbol{E_2}\boldsymbol{g}(\omega\boldsymbol{u} + \boldsymbol{\varepsilon}) + \frac{1}{2}\boldsymbol{\phi}_2^T\boldsymbol{\phi}_2(\boldsymbol{x},t) - \frac{1}{\mu_2}\widetilde{\boldsymbol{g}}\widetilde{\boldsymbol{\ell}}_{\boldsymbol{2}}\dot{\hat{\boldsymbol{\ell}}}_{\boldsymbol{2}}\end{aligned} \tag{48}$$

According to **Assumption 1** ($\omega_2$and $\varepsilon_2$are unknown), the upper and lower bounds of the fault parameters are defined as:

$$\underline{\boldsymbol{\omega}}_2 = inf(\boldsymbol{E_2}\boldsymbol{g}\omega),\ \boldsymbol{\vartheta}_2 = \frac{1}{\underline{\boldsymbol{\omega}}_{\boldsymbol{2}}} \tag{49}$$

$$\boldsymbol{\xi}_2 = sup(\boldsymbol{E_2}\boldsymbol{g}\varepsilon_2) \tag{50}$$

Take the Lyapunov function as:

$$\bar{V}_2 = V_2 + \frac{1}{2l_2}\underline{\boldsymbol{\omega}}_2\widetilde{\boldsymbol{\vartheta}}_2^T\widetilde{\boldsymbol{\vartheta}}_2 + \frac{1}{2r_2}\widetilde{\boldsymbol{\xi}}_2^T\widetilde{\boldsymbol{\xi}}_2 \tag{51}$$

where $l_2 > 0, r_2 > 0$ are design parameters. $\widetilde{\boldsymbol{\vartheta}}_{\boldsymbol{2}} = \boldsymbol{\vartheta_2} - \widehat{\boldsymbol{\vartheta}}_{\boldsymbol{2}}, \widetilde{\boldsymbol{\xi}}_{\boldsymbol{2}} = \boldsymbol{\xi_2} - \widehat{\boldsymbol{\xi}}_{\boldsymbol{2}}$ represent estimation errors. $\widehat{\boldsymbol{\vartheta}}_{\boldsymbol{2}}$ and $\widehat{\boldsymbol{\xi}}_{\boldsymbol{2}}$ are estimates of $\boldsymbol{\vartheta_2}$ and $\boldsymbol{\xi_2}$, respectively.

Differentiating $\bar{V}_2$ with respect to time yields:

$$\dot{\bar{V}}_2 = \dot{V}_2 - \frac{1}{l_2}\underline{\omega}_2\widetilde{\boldsymbol{\vartheta}}_{\boldsymbol{2}}\dot{\hat{\boldsymbol{\vartheta}}}_{\boldsymbol{2}} - \frac{1}{r_2}\widetilde{\boldsymbol{\xi}}_{\boldsymbol{2}}\dot{\hat{\boldsymbol{\xi}}}_{\boldsymbol{2}} \tag{52}$$

where$\ell_2 = \frac{||W_2||^2}{\tilde{g}}$. Intermediate control laws can be designed as:

$$\bar{\boldsymbol{u}} = k_2\boldsymbol{z}_2 + \frac{z_2\hat{\boldsymbol{\ell}}_2 S_2^T S_2}{4T} + \hat{\boldsymbol{\xi}}_2\, tanh\left(\frac{z_2}{a_2}\right) \tag{53}$$

Substituting (70) within (69) yields:

$$\dot{V}_2 \le \boldsymbol{z_2 E_2 g}\omega_2\boldsymbol{u} + \frac{\tilde{\boldsymbol{\ell}}_2\tilde{\boldsymbol{g}} z_2^T S_2^T S_2 z_2}{4T} + T + \frac{1}{2}\boldsymbol{\phi}_2^T\boldsymbol{\phi}_2 - \frac{\omega_2}{l_2}\tilde{\boldsymbol{\vartheta}}_2\dot{\hat{\boldsymbol{\vartheta}}}_2 + \boldsymbol{z_2}\bar{\boldsymbol{u}} - k_2\boldsymbol{z}_2^T\boldsymbol{z}_2 - \frac{\hat{\boldsymbol{\ell}}_2\tilde{\boldsymbol{g}} z_2^T S_2^T S_2 z_2}{4T} + \boldsymbol{\xi}_2\left[|\boldsymbol{z}_2| - \boldsymbol{z}_2\, tanh\left(\frac{z_2}{a_2}\right)\right] - \frac{1}{\mu_2}\tilde{\boldsymbol{g}}\tilde{\boldsymbol{\ell}}_2\dot{\hat{\boldsymbol{\ell}}}_2 + \frac{1}{r_2}\tilde{\boldsymbol{\xi}}_2\left[r_2\boldsymbol{z_2}\, tanh\left(\frac{z_2}{a_2}\right) - \dot{\hat{\boldsymbol{\xi}}}_2\right]$$

Designing adaptive laws for:

$$\dot{\hat{\boldsymbol{\ell}}}_2 = \frac{\mu_2 z_2^T S_2^T S_2 z_2}{4T} - \Gamma_2\hat{\boldsymbol{\ell}}_2 \tag{54}$$

$$\dot{\hat{\boldsymbol{\xi}}}_2 = r_2\boldsymbol{z}_2\, tanh\left(\frac{z_2}{a_2}\right) - b_2\hat{\boldsymbol{\xi}}_2 \tag{55}$$

$$\dot{\hat{\boldsymbol{\vartheta}}}_2 = l_2\boldsymbol{z_2}\bar{\boldsymbol{u}} - c_2\hat{\boldsymbol{\vartheta}}_2 \tag{56}$$

where $a_2 > 0$, $b_2 > 0$, $c_2 > 0$ are design parameters.

Substituting Eq. (54) ~ (66) it can be obtained:

$$\dot{V}_2 \le \boldsymbol{z_2 E_2 g}\omega_2\boldsymbol{u} + T + \frac{1}{2}\boldsymbol{\phi}_2^T\boldsymbol{\phi}_2 + \frac{\Gamma_2\tilde{\boldsymbol{g}}}{\mu_2}\tilde{\boldsymbol{\ell}}_2\hat{\boldsymbol{\ell}}_2 - \boldsymbol{\omega_2}\tilde{\boldsymbol{\vartheta}}_2\boldsymbol{z_2}\bar{\boldsymbol{u}} + \frac{\omega_2 c_2}{l_2}\tilde{\boldsymbol{\vartheta}}_2\hat{\boldsymbol{\vartheta}}_2 + \boldsymbol{z_2}\bar{\boldsymbol{u}} - k_2\boldsymbol{z}_2^T\boldsymbol{z}_2 + \boldsymbol{\xi_2}\left[|\boldsymbol{z_2}| - z_2\, tanh\left(\frac{z_2}{a_2}\right)\right] + \frac{b_2}{r_2}\tilde{\boldsymbol{\xi}}_2\hat{\boldsymbol{\xi}}_2$$

Designing the actual controller of PRC for:

$$\boldsymbol{u} = (\boldsymbol{g}^{-1})(-\boldsymbol{F}_1 - \boldsymbol{F}_2 - \frac{z_2\hat{\boldsymbol{\vartheta}}_2^2\bar{u}^2}{\sqrt{z_2^T\hat{\boldsymbol{\vartheta}}_2^T\bar{u}^T\boldsymbol{u}\hat{\boldsymbol{\vartheta}}_2 z_2 + \sigma_2^2}}) \tag{57}$$

where $\sigma_2 > 0$ is a design parameter, which serves to avoid the singular value problem.

Using Young's inequality, it can be obtained that:

$$\tilde{\boldsymbol{\ell}}_2\hat{\boldsymbol{\ell}}_2 = (\boldsymbol{\ell}_2 - \tilde{\boldsymbol{\ell}}_2)\tilde{\boldsymbol{\ell}}_2 \le -\frac{1}{2}\tilde{\boldsymbol{\ell}}_2^T\tilde{\boldsymbol{\ell}}_2 + \frac{1}{2}\boldsymbol{\ell}_2^T\boldsymbol{\ell}_2 \tag{58}$$

$$\tilde{\boldsymbol{\vartheta}}_2\hat{\boldsymbol{\vartheta}}_2 = (\boldsymbol{\vartheta}_2 - \tilde{\boldsymbol{\vartheta}}_2)\tilde{\boldsymbol{\vartheta}}_2 \le -\frac{1}{2}\tilde{\boldsymbol{\vartheta}}_2^T\tilde{\boldsymbol{\vartheta}}_2 + \frac{1}{2}\boldsymbol{\vartheta}_2^T\boldsymbol{\vartheta}_2 \tag{59}$$

$$\tilde{\boldsymbol{\xi}}_2\hat{\boldsymbol{\xi}}_2 = (\boldsymbol{\xi}_2 - \tilde{\boldsymbol{\xi}}_2)\tilde{\boldsymbol{\xi}}_2 \le -\frac{1}{2}\tilde{\boldsymbol{\xi}}_2^T\tilde{\boldsymbol{\xi}}_2 + \frac{1}{2}\boldsymbol{\xi}_2^T\boldsymbol{\xi}_2 \tag{60}$$

Substituting (58)~(60) with (57), and using Young's inequality and Lemma 1:

$$\dot{V}_2 \le -k_2\boldsymbol{z}_2^T\boldsymbol{z}_2 - \frac{1}{2\mu_2}\Gamma_2\tilde{\boldsymbol{g}}\tilde{\boldsymbol{\ell}}_2^T\tilde{\boldsymbol{\ell}}_2 - \frac{1}{2l_2}\omega_2 c_2\tilde{\boldsymbol{\vartheta}}_2^T\tilde{\boldsymbol{\vartheta}}_2 - \frac{1}{2r_2}b_2\tilde{\boldsymbol{\xi}}_2^T\tilde{\boldsymbol{\xi}}_2 + \frac{1}{2\mu_2}\Gamma_2\tilde{\boldsymbol{g}}\boldsymbol{\ell}_2^T\boldsymbol{\ell}_2 + \frac{1}{2l_2}\boldsymbol{\omega_2}c_2\boldsymbol{\vartheta}_2^T\boldsymbol{\vartheta}_2 + \frac{1}{2r_2}b_2\boldsymbol{\xi}_2^T\boldsymbol{\xi}_2 + T + \frac{1}{2}\boldsymbol{\phi}_2^T\boldsymbol{\phi}_2 + \boldsymbol{\omega_2\sigma_2} + 0.2785a_2\boldsymbol{\xi}_2 \tag{61}$$